Math 131
Midterm Examination 1 Solutions – February 8, 2009

**General Instructions:** You may use a simple calculator that is not graphing or programmable. You may have a 3x5 card, but no other notes.

**Part I (72 points):** For each of the following 12 multiple choice problems, mark your answer on the answer card. For Part I, only the answer on the card will be graded. Each problem is worth 6 points.

1. Compute

\[ \lim_{x \to 0} x^2 + 1 \]

(a) Undefined/doesn’t exist
(b) 0
(c) 1
(d) 2
(e) \( \sqrt{3} \)
(f) -1
(g) -2
(h) \(-\sqrt{3}\)

Since \( x^2 + 1 \) is continuous everywhere, we plug in \( 0^2 + 1 = 1 \). The answer is C.

2. Which of the following functions are continuous at \( x = 0 \)?

1. \( \sin \frac{1}{x} \)
2. \( \frac{1}{x^2} \)
3. \( \frac{1}{x^2 + 1} \)

(a) None of the above
(b) Only 1.
(c) Only 2.
(d) Only 1 and 2.
(e) Only 3.
(f) Only 1 and 3.
(g) Only 2 and 3.
(h) All of the above

The first two functions are not defined at $x = 0$, so certainly not continuous there. The third function is defined at $x = 0$, and rational functions are continuous everywhere that they are defined. The answer is E.

3. Find $\lim_{x \to 0^+} \cos \frac{1}{x}$.

(a) Undefined/doesn’t exist
(b) 0
(c) 1
(d) 2
(e) $\infty$
(f) -1
(g) -2
(h) $-\infty$

The function $\cos \frac{1}{x}$ oscillates infinitely as $x \to 0$, so the limit does not exist. The answer is A.

4. Compute $\lim_{x \to 3} \frac{3 - x}{\sqrt{x} - \sqrt{3}}$.

(a) Undefined/doesn’t exist
(b) 0
(c) 1
(d) -1
(e) $\sqrt{3}$
When we try plugging in 3, we get something of the form “0/0”. We cancel the zeros, using the conjugate technique from class:

\[
\lim_{x \to 3} \frac{3 - x}{\sqrt{x} - \sqrt{3}} \cdot \frac{\sqrt{x} + \sqrt{3}}{\sqrt{x} + \sqrt{3}} = \lim_{x \to 3} \frac{(3 - x) \cdot (\sqrt{x} + \sqrt{3})}{x - 3} = \lim_{x \to 3} -(\sqrt{x} + \sqrt{3}).
\]

Since \(-(\sqrt{x}+\sqrt{3})\) is continuous, we plug in 3 to get \(-2\sqrt{3}\). The correct answer is H.

Since sign errors are especially easy to make, I gave half credit for answer G.

5. Consider the function \(g(x) = \frac{x^2 - 1}{|x + 1|}\). Find \(\lim_{x \to -1^+} g(x)\).

Hint: Remember the piecewise definition of \(|x + 1|\)!

(a) Undefined/doesn’t exist
(b) 0
(c) 1
(d) 2
(e) \(\infty\)
(f) -1
(g) -2
(h) \(-\infty\)

We follow the hint, and write

\[
g(x) = \begin{cases} 
\frac{x^2-1}{x+1} & \text{if } x + 1 \geq 0 \\
\frac{x^2-1}{-(x+1)} & \text{if } x + 1 < 0 
\end{cases}
\]

\[
g(x) = \begin{cases} 
x - 1 & \text{if } x > -1 \\
1 - x & \text{if } x < -1 \\
\text{undefined} & \text{if } x = -1
\end{cases}
\]
Since the limit is as $x \rightarrow -1^+$, we plug into the function for $x \geq -1$, to get $-2$. The answer is G.

6. Consider the function $g(x) = \frac{x^2 - 1}{|x + 1|}$. Find $\lim_{x \to -1} g(x)$

(a) Undefined/doesn’t exist
(b) 0
(c) 1
(d) 2
(e) $\infty$
(f) -1
(g) -2
(h) $-\infty$

See the answer for Problem 5. The limit as $x \rightarrow -1^-$ we find by plugging into $1 - x$, for which we get 2. Since $2 \neq -2$, the left and right limits do not agree, and the (two-sided) limit is undefined.

The answer is A.

7. Let

$$f(x) = \begin{cases} 
-1 & \text{if } x < -5 \\
x^2 & \text{if } -5 \leq x < 0 \\
x^3 & \text{if } 0 \leq x \leq 5 \\
x^2 + 100 & \text{if } x > 5 
\end{cases}.$$

Find $\lim_{x \to 0} f(x)$.

(a) Undefined/doesn’t exist
(b) 0
(c) 1
(d) 2
(e) $\infty$
(f) -1
We calculate $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^3 = 0$, and $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 = 0$. Since the left and right limits agree, the two-sided limit is the same. The answer is B.

8. Let

$$f(x) = \begin{cases} 
-1 & \text{if } x < -5 \\
 x^2 & \text{if } -5 \leq x < 0 \\
x^3 & \text{if } 0 \leq x \leq 5 \\
x^2 + 100 & \text{if } x > 5
\end{cases}$$

the same as in the last problem. Where is $f$ continuous?

(a) Everywhere.
(b) Everywhere except $-5$, $0$, $1$, and $5$.
(c) Everywhere except $-5$, $0$, and $5$.
(d) Everywhere except $-5$, and $1$.
(e) Everywhere except $-5$.
(f) Everywhere except $-5$, and $0$.
(g) Everywhere except $5$.
(h) Everywhere except $0$.

The function is clearly continuous everywhere except possibly $-5$, $0$, and $5$. We check each point.

At $-5$: $\lim_{x \to -5^-} f(x) = \lim_{x \to -5^-} -1 = -1$, while $\lim_{x \to -5^+} f(x) = \lim_{x \to -5^+} x^2 = 25$. Since these disagree, the (two-sided) limit does not exist, so the function is not continuous at $-5$.

At $0$: The function is not defined at $0$, so not continuous at $0$.

At $5$: $\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} x^3 = 125$, while $\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} x^2 + 100 = 125$. Since the left and right limits agree, the two-sided limit is also $125$. Also $f(5) = 5^3 = 125$, the same as the limit. Thus the function is continuous at $5$. 

(g) -2
(h) $-\infty$
The answer is F. Since the behavior at 0 was a bit tricky, I gave half credit for E.

9. If you take the graph of \( f(x) = x^3 \), and first shift it left by 4 units, then expand it vertically by a factor of two, what function do you get?

(a) \( 2x^3 + 4 \)
(b) \( 2x^3 - 4 \)
(c) \( 2(x + 4)^3 \)
(d) \( 2(x - 4)^3 \)
(e) \( \frac{1}{2}(x + 4)^3 \)
(f) \( \frac{1}{2}(x - 4)^3 \)
(g) \( \frac{1}{2}x^3 + 4 \)
(h) \( \frac{1}{2}x^3 - 4 \)

Shifting left replaces \( x \) by \( x + 4 \). Expanding vertically multiplies by 2. The answer is C.

10. Find \( \lim_{x \to 0} \frac{x^2 - e^x \sin x}{\cos^2 x} \).

(a) Undefined/doesn’t exist
(b) 0
(c) 1
(d) 2
(e) \( \infty \)
(f) -1
(g) -2
(h) \( -\infty \)

Since the function is defined at 0, and built from \( x^2, e^x, \sin x, \) and \( \cos x \) by subtraction, multiplication, and division, it is continuous. Thus, we plug in \( x = 0 \): \( \frac{0^2-1\cdot0}{1^2} = 0 \). The answer is B.
11. Find \( \lim_{x \to 5} \frac{(3x - 15) \cdot \sin(x - 5)}{2(x - 5)^2} \)

(a) Undefined
(b) \(-\frac{3}{2}\)
(c) \(\frac{2}{3}\)
(d) \(\frac{9}{2}\)
(e) \(\frac{3}{2}\)
(f) \(\frac{9}{2}\)
(g) \(\frac{1}{3}\)
(h) \(-\frac{9}{2}\)

We rewrite as
\[
\lim_{x \to 5} \frac{3}{2} \cdot \frac{(x - 5) \sin(x - 5)}{(x - 5)^2} = \lim_{x \to 5} \frac{3}{2} \cdot \frac{\sin(x - 5)}{(x - 5)},
\]
and since \(\frac{\sin(x-5)}{x-5}\) is \(\frac{\sin x}{x}\) shifted to the right by 5, the limit \(\lim_{x \to 5} \frac{\sin(x-5)}{(x-5)}\)
is 1. Thus, our overall answer is \(\frac{3}{2}\), or E.

12. Which of the below is equivalent with the statement “\( f \) is continuous at \( a \)?”

(a) for all \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( |x - a| < \delta \implies |f(x) - L| < \epsilon \).
(b) for all \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( |x - a| < \delta \implies |f(x) - f(a)| < \epsilon \).
(c) for all \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( |x - a| < \delta \implies |f(x) - f(L)| < \epsilon \).
(d) for all \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( 0 < x - a < \delta \implies |f(x) - L| < \epsilon \).
(e) for all \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( 0 < x - a < \delta \implies |f(x) - f(a)| < \epsilon \).
(f) for all \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that \( 0 < x - a < \delta \implies |f(x) - f(L)| < \epsilon \).
(g) for all $\epsilon < 0$ there exists a $\delta < 0$ such that $|x - a| < \delta \implies |f(x) - L| < \epsilon$.

(h) Undefined/doesn’t exist

The answer A is essentially the definition of $\lim_{x \to a} f(x) = L$ (the definition of limit also has $0 < |x - a|$). However, we were asked about the definition of continuous function, which means $\lim_{x \to a} f(x) = f(a)$. When we plug $f(a)$ in for $L$, we get B.

Note: D is essentially the definition of right-sided limit, while E is the definition of right-sided continuity ($\lim_{x \to a^+} f(x) = f(a)$).

The answer is B. Two-thirds of the class put A – I gave half credit for this answer.

**Part II (28 points):** In each of the following problems, show your work clearly in the space provided. Partial credit will be given, and a correct answer without supporting work may not receive credit.

1. Rates of change and tangent lines
Let $f(x) = x^3$.

   (a) (2 points) Graph $f(x)$, and on the same axis, the straight line from $(0, 0)$ to $(h, h^3)$.
   (You might remember that this line is called a secant.)
(b) (4 points) Find the average rate of change in $f$ on the interval $[0, h]$.

The average rate of change is $\frac{\Delta y}{\Delta x}$, that is, the slope of the line in part (a). We calculate this to be

$$\frac{f(h) - f(0)}{h - 0} = \frac{h^3 - 0}{h - 0} = \frac{h^3}{h}.$$

(c) (5 points) By taking a limit, find the instantaneous rate of change of $f$ at $0$.

The instantaneous rate of change is $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, that is,

$$\lim_{h \to 0} \frac{h^3}{h} = \lim_{h \to 0} h^2 = 0.$$

2. Bounds

(a) (4 points) Find a bound for $|2 \sin \frac{1}{x^2} + 3 \cos \frac{1}{x^2}|$.

Following the handout, we break it up:

$$|2 \sin \frac{1}{x^2} + 3 \cos \frac{1}{x^2}| \leq |2 \sin \frac{1}{x^2}| + |3 \cos \frac{1}{x^2}|$$

$$= 2|\sin \frac{1}{x^2}| + 3|\cos \frac{1}{x^2}|$$

$$\leq 2 + 3 = 5,$$

where the last step is because $\sin *$ and $\cos *$ are bounded by 1. Our bound is 5.

(b) (5 points) Using the sandwich theorem, explain why $\lim_{x \to 0} x^3 \sin \frac{1}{x^2} = 0$.

Similarly to part (a), we know that

$$-1 \leq \sin \frac{1}{x^2} \leq 1,$$

hence

$$-|x^3| \leq x^3 \sin \frac{1}{x^2} \leq |x^3|.$$

Since $\lim_{x \to 0} x^3 = \lim_{x \to 0} |x^3| = \lim_{x \to 0} -|x^3| = 0$, $x^3 \sin \frac{1}{x^2}$ is sandwiched, and also has limit 0.
3. Let \( f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \).

(a) (3 points) For what values of \( x \) is \( f \) continuous?

All values. The function \( \frac{\sin x}{x} \) is continuous at all values except 0, and so \( f(x) \) is continuous at all such values. Zero is explained in part (b).

(b) (5 points) Explain carefully why the above \( f(x) \) is continuous or not continuous at \( x = 0 \).

For \( f \) to be continuous at 0, we must have \( \lim_{x \to 0} f(x) = f(0) \).

On the one hand, \( f(0) = 1 \). On the other, \( \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1 \).

These are equal, so the function is indeed continuous at 0.

The graph of \( f(x) \) is as follows: