Calculus I

Math 131 — Spring 2008

Exam 3, April 8

Name:                      Student-ID:

This exam contains sixteen problems. Problems 1 – 14 are multiple choice problems, which each count 5% towards your total score. Problem 15 and 16 will be hand-graded (with a possibility of partial credit) and count 15% each towards your total score.

Problem 1

On which intervals is \( f(x) = 4x^3 - x^4 \) increasing?

A) Only on \((-3, 0)\)
B) On \((-3, 0)\) and \((0, \infty)\)
C) Only on \((-\infty, 0)\) and \((0, 3)\)
D) Only on \((0, 3)\)
E) On \((-\infty, -\sqrt{3})\) and \((-\sqrt{3}, \sqrt{3})\)
F) Only on \((-\sqrt{3}, \sqrt{3})\)
G) Only on \((3, \infty)\)
H) This function is nowhere increasing.

\[
f'(x) = 12x^2 - 4x^3 = 4x^2(3-x)
\]

Critical Points: \( x = 0, x = 3 \)

\[
\begin{array}{c|cccc}
\text{Sign of } f'(x) & + & + & 0 & - - \\
\hline
\text{Sign of } f(x) & \nearrow & 0 & \searrow & 3 \downarrow
\end{array}
\]
Problem 2

List the x-coordinates of all the inflection points of

\[ y = \frac{1}{3}x^6 + \frac{1}{2}x^5 - 5x^4 + 4x + 8 \]

\[ y' = 2x^5 + \frac{5}{2}x^4 - 20x^3 + 4 \]

\[ y'' = 10x^4 + 10x^3 - 60x^2 \]

\[ = 10x^2(x^2 + x - 6) \]

\[ = 10x^2(x + 3)(x - 2) \]

\[ y'' = 0 \quad \text{AT} \quad x = 0, \quad x = -3, \quad x = 2 \]

\[ + + 0 -- -6 -- 0 -- -6 + + \]

\[ -3 \quad 0 \quad 2 \]

\( x = 0 \) is NO INFLECTION POINT

(NO SIGN CHANGE THERE)
Problem 3

Compute

\[ \lim_{x \to 0} \frac{x - \sin x}{x^3} = \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \]

\[ = \lim_{x \to 0} \frac{\sin x}{6x} \]

\[ = \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6} \]

(*L'Hôpital's Rule, Three Times*)

\[ (\text{OR: } \lim_{x \to 0} \frac{\sin x}{6x} = \frac{1}{6} \cdot \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{6} \cdot 1 = \frac{1}{6}) \]

A) -1
B) $-\frac{1}{3}$
C) $-\frac{1}{6}$
D) 0
E) $\frac{1}{6}$
F) $\frac{1}{3}$
G) 1
H) 6
Problem 4

Find the absolute minimum value and absolute maximum value of the function \( f(x) = x - 2\sin x \) on the interval \([0, \pi]\).

\[
\min, \max = (\ldots, \ldots)
\]

\( \frac{d}{dx}(x) = 1 - 2\cos x = 0 \)

\[
\cos x = \frac{1}{2}
\]

\( x \in [0, \pi], \) so \( x = \frac{\pi}{3} \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{\pi}{3} )</td>
<td>( \frac{\pi}{3} - 2\cdot\frac{\sqrt{3}}{2} = \frac{\pi}{3} - \sqrt{3} \approx -0.68 )</td>
</tr>
<tr>
<td>( \pi )</td>
<td>( \pi )</td>
</tr>
</tbody>
</table>

\( \text{MIN} \)

A) \( \left(\frac{\pi}{3}, 0\right) \)
B) \( (\pi, \sqrt{3}) \)
C) \( (0, \frac{\pi}{3} - \sqrt{3}) \)
D) \( (\pi - \sqrt{3}, \frac{\pi}{3}) \)
E) \( (\frac{\pi}{3}, 0) \)
F) \( (0, \pi) \)
G) \( (\frac{\pi}{3} - \sqrt{3}, \pi) \)
H) \( (0, \frac{\pi}{3}) \)
Problem 5

If $f$ is a differentiable function on the interval $[3, 7]$ and if $f(3) = 1$ and $f(7) = 4$, then the Mean Value Theorem says that there is some number $c$ in between 3 and 7 such that $f'(c) = \cdots$

Calculate $f'(c)$ in this situation.

\[ f'(c) = \frac{f(7) - f(3)}{7 - 3} = \frac{4 - 1}{7 - 3} = \frac{3}{4} \]

A) $\frac{3}{4}$
B) $\frac{4}{3}$
C) $\frac{1}{4}$
D) $\frac{3}{7}$
E) $\frac{1}{2}$
F) 4
G) 0
H) 1
Problem 6

The product of two positive numbers \( x \) and \( y \) equals 24. Find the smallest number that their sum can be.

\[ xy = 24 \implies y = \frac{24}{x} \]

A) \( \sqrt{24} \)
B) 6
C) \( 2\sqrt{12} \)
D) 8
E) \( 2\sqrt{24} \)
F) 11
G) 12
H) 24

\[ S(x) = x + \frac{24}{x} \quad \text{Domain : } (0, \infty) \]

\[ S'(x) = 1 - \frac{24}{x^2} = 0 \]

\[ x^2 = 24 \]

\[ x = \sqrt{24} \quad \text{OR} \quad x = -\sqrt{24} \]

\[ S'(x) \]

\[ \begin{array}{cccc}
\downarrow & \downarrow & \downarrow \\
0 & \sqrt{24} & \infty
\end{array} \]

So \( x = \sqrt{24} \), AND \( y = \frac{24}{\sqrt{24}} = \sqrt{24} \)

Then \( x + y = \sqrt{24} + \sqrt{24} = 2\sqrt{24} \)
Problem 7

Find all vertical asymptotes of

\[ f(x) = \frac{x^2 - 4x + 3}{x^2 - 5x + 6} = \frac{(x-3)(x-1)}{(x-3)(x-2)} = \frac{x-1}{x-2} \quad (\text{if } x \neq 3) \]

\[ \lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x-1}{x-2} = \frac{3-1}{3-2} = 2 \]

\[ \lim_{x \to 2}^+ f(x) = \lim_{x \to 2}^+ \frac{x-1}{x-2} = \infty \]

\[ \lim_{x \to 2}^- f(x) = -\infty \]

A) \( x = 1 \)

B) \( x = 2 \)

C) \( x = 2, x = 3 \)

D) \( x = 1, x = 3 \)

E) \( x = 1, x = 2 \)

F) \( x = 3 \)

G) \( x = 1, x = 2, x = 3 \)

H) There are no vertical asymptotes.
Problem 8

If a cylinder has height \( h \), and the radius of the base circle is \( r \), then the volume equals \( \pi r^2 h \), and the total surface area equals \( 2\pi r h + 2\pi r^2 \).

For a cylinder with a volume of 16 cm\(^3\), determine the radius \( r \) (in cm) which minimizes the total surface area.

\[
\pi r^2 h = 16 \quad \implies \quad h = \frac{16}{\pi r^2}
\]

A) \( \frac{\sqrt{2}}{2} \)

B) \( \frac{\sqrt{2}}{3} \)

C) \( \frac{2\pi}{3} \)

D) \( \frac{2}{\pi} \)

E) \( 2\sqrt{2} \)

F) \( \frac{2}{\sqrt{3}} \)

G) \( 3\sqrt{\pi} \)

H) \( \frac{\pi}{\sqrt{3}} \)

\( \text{MINIMIZE} \quad A(r) = 2\pi rh + 2\pi r^2 \)

\( = 2\pi r \cdot \frac{16}{\pi r^2} + 2\pi r^2 \)

\( = \frac{32}{r} + 2\pi r^2 \)

\( A'(r) = -\frac{32}{r^2} + 4\pi r = 0 \)

\( 4\pi r = \frac{32}{r^2} \)

\( r^3 = \frac{32}{4\pi} = \frac{8}{\pi} \)

\( r = \frac{\sqrt[3]{8}}{\sqrt[3]{\pi}} = \frac{2}{\sqrt[3]{\pi}} \)

\( A'(r) \) --- 0 --- \( \frac{2}{\sqrt[3]{\pi}} \) ---

\( \text{LOCAL MIN AT} \quad x = \frac{2}{\sqrt[3]{\pi}} \)
Problem 9

\[ \int (5x^2 - 17x^{-\frac{1}{2}} + e^x) \, dx = \ldots \]

\[ = \frac{1}{3} \cdot 5 \cdot x^3 - \frac{17}{\sqrt{2}} \cdot x^{\frac{3}{2}} + e^x + C \]

\[ = \frac{5}{3} x^3 - 34 \cdot x^{\frac{1}{2}} + e^x + C \]

(A) \( \frac{5}{3} x^3 - 34x^{\frac{1}{2}} + e^x + C \)
(B) \( \frac{5}{3} x^3 - 34x^{\frac{1}{2}} - e^x + C \)
(C) \( \frac{5}{3} x^3 - \frac{17}{2} x^{\frac{3}{2}} + e^x + C \)
(D) \( \frac{5}{3} x^3 - \frac{17}{2} x^{\frac{3}{2}} - e^x + C \)
(E) \( 5x^3 - 34x^{\frac{1}{2}} + e^x + C \)
(F) \( 5x^3 - 34x^{\frac{1}{2}} - e^x + C \)
(G) \( 5x^3 - \frac{17}{2} x^{\frac{3}{2}} + e^x + C \)
(H) \( 5x^3 - \frac{17}{2} x^{\frac{3}{2}} - e^x + C \)
Problem 10

Calculate the limit:

\[
\lim_{x \to \infty} \frac{x^2 \sqrt{4 - e^{-x}}}{-2x^2 + 5x - 3}
\]

\[
= \lim_{x \to \infty} \frac{x^2}{-2x^2 + 5x - 3} \cdot \sqrt{4 - e^{-x}}
\]

\[
= \lim_{x \to \infty} \frac{1}{-2 + \frac{5}{x} - \frac{3}{x^2}} \cdot \sqrt{4 - e^{-x}}
\]

\[
= \frac{1}{-2 + 0 + 0} \cdot \sqrt{4 - 0} = \frac{2}{-2} = -1
\]

\[\text{(Note: } \lim_{x \to \infty} e^{-x} = 0)\]

A) \(\infty\)
B) \(-\infty\)
C) 0
D) \(-1\)
E) 1
F) \(-\frac{1}{2}\)
G) \(-2\)
H) \(-\frac{\sqrt{3}}{2}\)
Problem 11

On which intervals is $f(x) = (x^2 - 4x + 2)e^x$ concave down?

A) $(-\infty, 1 - \sqrt{3})$ and $(1 + \sqrt{3}, \infty)$
B) $(1 - \sqrt{3}, 1 + \sqrt{3})$
C) $(-1, 1)$
D) $(-2, 2)$
E) $(-\infty, -2)$ and $(2, \infty)$
F) $(-\infty, 2 - \sqrt{2})$ and $(2 + \sqrt{2}, \infty)$
G) $(2 - \sqrt{2}, 2 + \sqrt{2})$
H) $(-\sqrt{2}, \sqrt{2})$

\[ f'(x) = (2x-4)e^x + (x-4x+2)e^x = (x^2 - 2x - 2)e^x \]
\[ f''(x) = (2x-2)e^x + (x^2 - 2x - 2)e^x = (x^2 - 4)e^x \]

\[ f''(x) = 0 \quad \Rightarrow \quad x = \pm 2 \]

\[ f''(x) \begin{array}{ccccccc}
+ & + & 0 & - & - & 0 & + & + \\
\hline
-2 & 2
\end{array} \]
Problem 12

\[
\lim_{t \to 1} \frac{e^{2t-2} - \cos(1 - t) - 2t + 2}{t^2 - 2t + 1} = \ldots
\]

\[
= \lim_{t \to 1} \frac{2 e^{2t-2} - \sin(1-t) - 2}{2t - 2}
\]

\[
= \lim_{t \to 1} \frac{4 e^{2t-2} + \cos(1-t)}{2} = \frac{4 + 1}{2} = \frac{5}{2}
\]

A) -1
B) -\frac{1}{2}
C) 0
D) \frac{1}{2}
E) 1
F) \frac{3}{2}
G) 2
H) \frac{5}{2}
Problem 13

\( x - 1 < -2 \) \ OR \( x - 1 > 2 \)
\( x < -1 \) \ OR \( x > 3 \)

What are the \( x \)-values of the critical points of \( f(x) = \sqrt{4 - (x - 1)^2} \)?

\[
\frac{d}{dx} f(x) = \frac{\frac{1}{2} \left(4 - (x - 1)^2\right)^{-\frac{1}{2}} \cdot (-2(x - 1))}{\sqrt{4 - (x - 1)^2}}
\]

A) \( x = -2, x = 1, x = 2 \)
B) \( x = -2, x = 2 \)
C) \( x = 1 \)
D) \( x = 1, x = 2, x = 3 \)
E) \( x = -1, x = 3 \)
F) \( x = 1 \)
G) \( x = -1, x = 1, x = 3 \)

H) There are no critical points.
Problem 14

Consider the function \( f(x) = x + \frac{4}{(x-3)^2} \) with domain \((2, 5] \).
Does \( f(x) \) have an absolute minimum value? If so, where does it occur?
What about an absolute maximum value?

\[ \lim_{{x \to 2^+}} f(x) = +\infty \rightarrow \text{NO MAX} \]

- A) min at \( x = 5 \), no max
- B) min at \( x = 5 \), max at \( x = 3 \)
- C) min at \( x = 4 \), no max
- D) min at \( x = 4 \), max at \( x = 3 \)
- E) max at \( x = 2 \), no min
- F) max at \( x = 2 \), min at \( x = 4 \)
- G) max at \( x = 3 \), no min
- H) max at \( x = 2 \), min at \( x = 5 \)

\[
\begin{align*}
\frac{d^2}{dx^2} f(x) & = 1 + 4 \cdot (-2) \frac{1}{(x-2)^3} \\
& = 1 - \frac{8}{(x-2)^3}
\end{align*}
\]

\[
\frac{d}{dx} f(x) = 0 \Leftrightarrow (x-2)^3 = 8 \\
\Leftrightarrow x-2 = 2 \\
\Leftrightarrow x = 4
\]

\[
\begin{array}{cccccc}
2 & 4 & 5 \\
\hline
\frac{d}{dx} f(x) & - & - & - & 0 & + \\
\hline
\end{array}
\]

Local min at \( x = 4 \), but this is also an absolute min since \( f(x) \) is decreasing on \((2, 4]\) and increasing on \([4, 5] \).
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Written problem - Show your work

You can get partial credit for this one, up to a maximum of 14 points + 1 point for writing down your name and student ID legibly!

Problem 15

Sketch the graph of $f(x) = x^4 - x^3$.
(Point out at least the following by calculation: critical points, where is $f$ increasing/decreasing, inflection points, where is $f$ concave up/down.
Then make a sketch of the graph:
the parts where $f$ is increasing/decreasing/concave up/concave down should be clearly visible. Point out where the critical points and/or inflection points are located.)

$$f'(x) = 4x^3 - 3x^2 = x^2(4x-3)$$
Critical Points: $x = 0, x = \frac{3}{4}$

$$f''(x) = 12x^2 - 6x = 6x(2x-1)$$

$f''(x) = 0$ at $x = 0, x = \frac{1}{2}$

Concave Up on $(-\infty, 0)$ and $\left(\frac{1}{2}, \infty\right)$.
Concave Down on $(0, \frac{1}{2})$

$$f(0) = 0$$
$$f\left(\frac{1}{2}\right) = -0.0625$$
$$f\left(\frac{3}{4}\right) \approx -0.11$$

DECREASING ON $(-\infty, 0)$ AND $(0, \frac{3}{4})$
INCREASING ON $\left(\frac{3}{4}, \infty\right)$

Inflation Points: $x = 0, x = \frac{1}{2}$
Written problem - Show your work

You can get partial credit for this one, up to a maximum of 14 points + 1 point for writing down your name and student ID legibly!

Problem 16

Approximate a root of \( f(x) = 2x + \ln x \) using Newton's Method, with initial value \( x_0 = 1 \). Do this by computing the first three approximations \( x_1, x_2 \) and \( x_3 \), using at least 3 decimals when rounding off numbers in your calculations.

\[
f'(x) = 2 + \frac{1}{x}
\]

\[
x_{n+1} = x_n - \frac{2x_n + \ln(x_n)}{2 + \frac{1}{x_n}}
\]

\[
x_0 = 1
\]

\[
x_1 = 1 - \frac{2}{3} = \frac{1}{3}
\]

\[
x_2 = \frac{1}{3} - \frac{\frac{2}{3} + \ln(\frac{1}{3})}{2 + \frac{1}{\frac{1}{3}}} \approx 0.4197
\]

\[
x_3 = 0.4197 - \frac{2 \cdot 0.4197 + \ln(0.4197)}{2 + \frac{1}{0.4197}} \approx 0.4263
\]