This exam contains sixteen questions. The first fourteen are multiple choice questions and count for five points each. There is no partial credit on these questions, so read each question carefully, check your arithmetic and make sure that you have marked the answer you intended to mark. The last two questions, which are each worth fifteen points, require written answers, and some partial credit might be given. However, no credit will be given for information that is not germane to the problem at hand. Please make sure to write your name and student ID number on the pages that include your answers to the last two questions. In fact, you will get one point on each of these two questions for writing your name and ID number legibly.
1. Suppose \( \frac{dy}{dx} = \frac{1}{y} \)

and \( y(0) = 1 \). Find \( y(4) \).

(a) 0
(b) \( \pm 2 \)
(c) \( \pm 3 \)
(d) \( \pm 4 \)
(e) \( e^d \)
(f) \( \ln 4 \)
(g) \( 1/4 \)
(h) \( -1/4 \)

\[
\frac{dy}{dx} = \frac{1}{y}
\]

\[ \therefore y \cdot dy = dx \]

\[ \therefore \frac{1}{2} y^2 = x + c \]

\[ y(0) = 1 \Rightarrow \frac{1}{2} = 0 + c \Rightarrow c = \frac{1}{2} \]

\[ y^2 = 2x + 1 \]

\[ \therefore y = \pm \sqrt{2x+1} \]

\[ \therefore y(4) = \pm \sqrt{9} = \pm 3 \]
2. Solve the differential equation

\[ \frac{dy}{dx} = xe^{x^2-y} \]

with initial condition \( y(0) = 0 \).

(a) \( e^y = e^{x^2} - 1 \)
(b) \( e^y = \frac{1}{2} e^{x^2} + \frac{1}{2} \)
(c) \( e^y = \frac{1}{2} e^{x^2} \)
(d) \( e^y = e^{x^2} \)
(e) \( e^{-y} = e^{x^2} - 1 \)
(f) \( e^{-y} = \frac{1}{2} e^{x^2} + \frac{1}{2} \)
(g) \( e^{-y} = \frac{1}{2} e^{x^2} \)
(h) \( e^{-y} = e^{x^2} \)

\[
e^y \, dy = xe^{x^2} \, dx
\]

\[
e^y = \int xe^{x^2} \, dx
\]

\[
= \frac{1}{2} \int e^u \, du
\]

\[
= \frac{1}{2} e^u + c
\]

\[
= \frac{1}{2} e^{x^2} + c
\]

\( y(0) = 0 \Rightarrow 0 = \frac{1}{2} + c \Rightarrow c = \frac{1}{2} \)

\( \therefore e^y = \frac{1}{2} e^{x^2} + \frac{1}{2} \)
3. Let $a_n = \frac{(-1)^n}{n}$. The sequence $\{a_n\}$

(a) converges to $-1$
(b) converges to 0
(c) converges to 1
(d) converges because it is an alternating series
(e) diverges because it is an alternating series
(f) diverges to $\infty$
(g) diverges to $-\infty$
(h) diverges but not to $\infty$ or $-\infty$
4. Let \( a_n = \left( \frac{n+3}{n} \right)^n \). Compute \( \lim_{n \to \infty} a_n \).

(a) 0
(b) 1
(c) \( e \)
(d) 3
(e) \( \pi \)
(f) \( e^3 \)
(g) \( \pi^3 \)
(h) \( \infty \)

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \left( 1 + \frac{3}{n} \right)^n = e^3 \quad (\text{Theorem 5, p. 417})
\]
5. Compute the sum of the series
\[ \sum_{n=2}^{\infty} \frac{7}{4^n} \]

(a) 0  
(b) \infty  
(c) 7/12  
(d) 28/3  
(e) 7/3  
(f) 21/4  
(g) 21/16  
(h) 7

\[ \frac{7}{4^2} + \frac{7}{4^3} + \frac{7}{4^4} + \cdots = \frac{7}{16} \left( 1 + \frac{1}{4} + \frac{1}{4^2} + \cdots \right) \]
\[ = \frac{7}{16} \left( \frac{1}{1 - \frac{1}{4}} \right) \]
\[ = \left( \frac{7}{16} \right) \left( \frac{4}{3} \right) \]
\[ = \frac{7}{12} \]
6. Compute the sum of the series

\[
\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}}
\]

(a) 0
(b) \(\infty\)
(c) \(-\infty\)
(d) 1/3
(e) 1/2
(f) 1/6
(g) \(-1/3\)
(h) 4/5

\[
\sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^{n-1} - \sum_{n=1}^{\infty} \left( \frac{1}{6} \right)^{n-1}
\]

\[
= \frac{1}{1 - \frac{1}{2}} - \frac{1}{1 - \frac{1}{6}}
\]

\[
= 2 - \frac{6}{5} = \frac{4}{5}
\]
7. Compute the sum of the series

\[ \sum_{n=1}^{\infty} \left( \cos \left( \frac{1}{n} \right) - \cos \left( \frac{1}{n+1} \right) \right) \]

(a) diverges to \( \infty \)
(b) diverges to \(-\infty\)
(c) diverges but not to \(-\infty\) or \(\infty\)
(d) \(-1\)
(e) \(-1 + \cos 1\)
(f) 0
(g) 1
(h) \cos 1

Telescoping Series:

\[ S_n = \left( \cos 1 - \cos \frac{1}{2} \right) + \left( \cos \frac{1}{2} - \cos \frac{1}{3} \right) + \cdots + \left( \cos \frac{1}{n} - \cos \frac{1}{n+1} \right) \]

\[ = \cos 1 - \cos \frac{1}{n+1} \]

\[ \therefore \lim_{n \to \infty} S_n = \cos 1 - \cos 0 = (\cos 1) - 1 \]
8. Which of the following three series is convergent?

\[
(A) \sum_{n=1}^{\infty} \frac{n+2}{n^{3/2}} \quad (B) \sum_{n=1}^{\infty} \frac{3}{n^4 - 2n^2} \quad (C) \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4 + 5}}
\]

(a) A only
(b) B only
(c) C only
(d) A and B only
(e) A and C only
(f) B and C only
(g) all
(h) none

\[\frac{n+2}{n^{3/2}} \sim \frac{1}{n^{1/2}} \Rightarrow \text{So use The Comparison Test (b) with } d_n = \frac{1}{n}\]

\[\sum \frac{1}{n^{1/2}} \text{ diverges by p-series test } \quad p = \frac{1}{2} \leq 1\]

\[\frac{3}{n^4 - 2n^2} \sim \frac{3}{n^4} \Rightarrow \text{So use the Limit Comparison Test, part 1} \]

\[3 \sum \frac{1}{n^4} \text{ converges by p-series test } \quad p = 4 > 1\]

\[\frac{n}{\sqrt{n^4 + 5}} \sim \frac{n}{n^2} = \frac{1}{n} \Rightarrow \text{So use Limit Comparison Test, part 1} \]

\[\sum \frac{1}{n} \text{ diverges by p-series test } \quad p = 1 \leq 1\]
9. Apply the Ratio Test to

\[ \sum_{n=1}^{\infty} \frac{2^n}{n!} \]

Find \( \rho \) and, if possible, decide whether the series converges or diverges.

(a) \( \rho = 0 \) and the series converges by the Ratio Test.
(b) \( \rho = 0 \) and the series diverges by the Ratio Test.
(c) \( \rho = 0 \) and the Ratio Test fails.
(d) \( \rho = 2 \) and the series converges by the Ratio Test.
(e) \( \rho = 2 \) and the series diverges by the Ratio Test.
(f) \( \rho = 2 \) and the Ratio Test fails.
(g) \( \rho = \infty \) and the series diverges by the Ratio Test.
(h) \( \rho = \infty \) and the Ratio Test fails.

\[
\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \\
= \frac{2}{(n+1)!} \cdot n! \\
= \frac{2}{n+1}
\]

\[ \therefore \rho = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{2}{n+1} = 0 < 1 \]

\[ \therefore \text{series converges} \]
10. Apply the Root Test to

\[ \sum_{n=1}^{\infty} (1 - 2/n)^n \]

Find \( \rho \) and, if possible, decide whether the series converges or diverges.

(a) \( \rho = e^{-2} \) and the series converges by the Root Test.

(b) \( \rho = e^{-2} \) and the series diverges by the Root Test.

(c) \( \rho = e^{-2} \) and the Root Test fails.

(d) \( \rho = 1 \) and the series converges by the Root Test.

(e) \( \rho = 1 \) and the series diverges by the Root Test.

(f) \( \rho = 1 \) and the Root Test fails.

(g) \( \rho = \infty \) and the series diverges by the Root Test.

(h) \( \rho = \infty \) and the Root Test fails.

\[
\rho = \lim_{n \to \infty} a_n^{1/n} = \lim_{n \to \infty} (1 - \frac{2}{n}) = 1
\]

\[ \therefore \text{Root Test Fails} \]
11. Which of the following three alternating series is convergent?

\[
(A) \sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n+2} \quad (B) \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}} \quad (C) \sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}
\]

(a) A only  
(b) B only  
(c) C only  
(d) A and B only  
(e) A and C only  
(f) B and C only  
(g) all  
(h) none

(A) diverges by \(n\)th term test  
(B) converges by the Alternating Series Test  
(C) converges by the Alternating Series Test
12. Which of the following three alternating series is absolutely convergent?

\[ (A) \sum_{n=1}^{\infty} (-1)^n \frac{n + 1}{n + 2} \quad (B) \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}} \quad (C) \sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n} \]

(a) A only  
(b) B only  
(c) C only  
(d) A and B only  
(e) A and C only  
(f) B and C only  
(g) all  
(h) none

(A) diverges by \( n \)th term test

\[ \sum \frac{1}{\sqrt{n}} \] diverges by \( p \)-series test  
\[ p = \frac{1}{2} \leq 1 \]

\[ \sum \frac{\ln n}{n} \] diverges by comparison test with \( d_n = \frac{1}{n} \)  
(or by integral test)
13. Find the radius of convergence for the series

\[ \sum_{n=1}^{\infty} \frac{3nx^n}{n + 2} \]

(a) \( R = 0 \)
(b) \( R = 1 \)
(c) \( R = 1/2 \)
(d) \( R = 3 \)
(e) \( R = 3/2 \)
(f) \( R = -1 \)
(g) \( R = x \)
(h) \( R = \infty \)

\[ \frac{a_{n+1}}{a_n} = \frac{3(n+1)x^{n+1}}{n+3} \cdot \frac{n+2}{3nx^n} \]

\[ = \frac{(n+1)(n+2)}{n(n+3)} \cdot x \]

\[ \therefore \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x| \]

The series converges absolutely if \( |x| < 1 \) and diverges if \( |x| > 1 \)

\[ \therefore R = 1 \]
14. For what values of $x$ does the series
\[ \sum_{n=1}^{\infty} \frac{(x-2)^n}{10^n} = \sum_{n=1}^{\infty} \left(\frac{x-2}{10}\right)^n \]
converge absolutely?
(a) $-1 < x < 1$
(b) $-1 < x < 3$
(c) $0 < x < 3$
(d) $0 < x < 10$
(e) $-8 < x < 12$
(f) $2 < x < 12$
(g) $-12 < x < 12$
(h) $-10 < x < 10$

Apply Geometric Series Test:
The series converges absolutely if
\[ \left| \frac{x-2}{10} \right| < 1 \]
and diverges otherwise
-10 < x - 2 < 10
-8 < x < 12
15. The series
\[ \frac{1}{1 + t} = 1 - t + t^2 - t^3 + \ldots + (-t)^n + \ldots \]
converges on the open interval \(-1 < t < 1\). Find the power series for \(\ln(1 + x)\) and compute its radius of convergence. Show your work.

\[
\ln(1 + x) = \int_0^x \left( 1 - t + t^2 - t^3 + \ldots + (-t)^n + \ldots \right) \, dt
\]

\[
= \left[ t - \frac{t^2}{2} + \frac{t^3}{3} + \ldots + \frac{(-t)^{n+1}}{n+1} + \ldots \right]_0^x
\]

\[
= x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{(-x)^{n+1}}{n+1} + \ldots
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} x^{n+1}
\]

\[ R = 1 \, (\text{using either Ratio or Root Test}) \]
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Name: 

Student ID: 

16. Use the Integral Test to decide whether

$$\sum_{n=2}^{\infty} \frac{4}{n \ln(n)}$$

converges or diverges. Show your work. Remember that you must show clearly that all conditions of the Integral Test are satisfied.

Set $$f(x) = \frac{4}{x \ln(x)}$$

$$f$$ is continuous, positive, and decreasing

for all $$x > 2$$.

$$\int_{2}^{\infty} \frac{4}{x \ln(x)} \, dx$$

either both converge or

both diverge.

$$\int_{2}^{\infty} \frac{4}{x \ln(x)} \, dx = \int_{\ln(2)}^{\infty} \frac{4}{u} \, du$$

$$u = \ln x$$

$$du = \frac{1}{x} \, dx$$

$$= \lim_{b \to \infty} 4 \ln |u| \bigg|_{\ln 2}^{b}$$

$$= \lim_{b \to \infty} 4(b - \ln 2)$$

$$= \infty.$$