Differential equations

Math 217 — Spring 2008

In-term exam February 5th.

This exam contains fourteen problems numbered 1 through 14. Problems 1 – 13 are multiple choice problems, which each count 6% of your total score. Problem 14 will be hand-graded and counts 22% of your total score.

Problem 1
Solve
\[ \frac{dy}{dx} = \frac{y}{x}, \quad y(1) = 2. \]
What is \( y(2) \)?
A) 0    B) 1    C) 2    D) 3    E) 4    F) 5
Problem 2

Identify the differential equation whose slope field looks like this:

\[ y' = -y - \sin x \quad \text{B) } y' = x + y \quad \text{C) } y' = x - y \quad \text{D) } y' = y - x + 1 \]
\[ \text{E) } y' = x - y + 1 \quad \text{F) } y' = \sin x + \sin y \]
Problem 4

A population $P(t)$ of sparrows is known to satisfy the natural growth equation

$$\frac{dP}{dt} = kP.$$ 

Find $k$ if $P(0) = 217$ and $P(2.5) = 2008$. Pick the closest answer.

A) 0.8  B) 0.9  C) 1.0  D) 1.1  E) 1.2  F) 1.3
Problem 5

Find the solution of the initial value problem

\[ y' + \frac{2}{t} y = \frac{\cos t}{t^2}, \quad y(\pi) = 0, \quad t > 0. \]

A) \( \frac{\sin t}{t} \)  \quad B) \( \frac{\sin(t^2)}{t} \)  \quad C) \( \frac{\sin t}{t^2} \)  \quad D) \( \frac{\sin(t^2)}{t^2} \)  \quad E) \( \frac{\cos(t^2)}{t} \)  \quad F) \( \frac{\cos t}{t^2} \)
Problem 6

At time $t = 0$, a tank contains 50 lb. of salt dissolved in 100 gallons of water. Assume that water containing $\frac{1}{4}$ lb. of salt per gallon is entering the tank at a rate of 3 gallons per minute and that the well-stirred mixture is draining from the tank at the same rate. Find the limiting amount of salt that is present after a very long time.

A) 100 lb.  B) 50 lb.  C) 25 lb.  D) 0.25 lb.  E) 3 lb.  F) 15 lb.
Problem 7

Which one of the following equations is exact?

A) \((2x + 4y) + (2x - 2y)y' = 0\)

B) \((x \ln(y) + xy) \, dx + (y \ln(x) + xy) \, dy = 0\)

C) \((x + 2y) + (2x + 3y) \frac{dy}{dx} = 0\)

D) \((x - 2y) + (2x - 3y) \frac{dy}{dx} = 0\)

E) \((3xy + y^2) + (x^2 + xy)y' = 0\)

F) \(e^x \sin y - (3x - e^x \sin y)y' = 0\)
Problem 8

Find the solution to the initial value problem

\[ 2xyy' = 4x^2 + 3y^2, \quad y(1) = -1. \]

A) \( y = -5x^3 + 4x \)  
B) \( y = -5x^3 + 4x^2 \)  
C) \( y = -5x^2 + 4x^3 \)

D) \( y^2 = 5x^3 - 4x \)  
E) \( y^2 = 5x^3 - 4x^2 \)  
F) \( y^2 = 5x^2 - 4x^3 \)
Problem 9

Consider the differential equation

\[ \frac{dx}{dt} = x(x + 1)(x - 10). \]

Which of the following statements are true?

I) \( x = 0 \) is a stable critical point.

II) \( x = 1 \) is an unstable critical point.

III) \( x = 10 \) is a stable critical point.

A) Only I  B) Only II  C) Only III  D) I and II  E) I and III  F) II and III
Problem 10

Suppose that the logistic equation $x' = x(5 - x)$ models a population $x(t)$ of fish in a lake after $t$ months during which no fishing occurs. Now suppose that, because of fishing, fish are removed from the lake at the rate of $3x$ fish per month. What is the limiting population?

A) 0  B) 2  C) 4  D) 5  E) 6  F) 8
Problem 11

A baseball that falls towards the ground satisfies

\[ \frac{dv}{dt} = -\rho v - g, \]

where \( \rho \) is some positive constant and \( g = 32 \text{ft/s}^2 \). Assume that \( v(0) = 0 \) and \( \lim_{t\to\infty} v(t) = -100 \). Find \( v(1) \), and pick the closest answer.

A) -21  B) -24  C) -27  D) -30  E) -33  F) -36
Problem 12

Solve

\[ y' - 2xy = 2xy^2, \quad y(0) = 1. \]

A) \( 2e^{-x^2} - 1 \) \quad B) \( \frac{1}{2e^{-x^2} - 1} \) \quad C) \( e^{x^2} \) \quad D) \( e^{-x^2} \) \quad E) \( 2 - e^{x^2} \) \quad F) \( \frac{1}{2 - e^{x^2}} \)
Problem 13

Find the limit $\lim_{t \to \infty} y(t)$ where $y(t)$ is the solution to the initial value problem

$$y' + 2y = 4, \quad y(0) = 1.$$ 

A) 0  B) 1  C) 2  D) 3  E) 4  F) The limit does not exist
The following problem will be hand-graded. To earn full credit you need to justify your answers.

**Problem 14**

Consider a cascade of two tanks. Tank 1 initially contains 100 gallons of pure ethanol and tank 2 initially contains 100 gallons of pure water. Pure water flows into tank 1 at 10 gallons per minute, and the other two flow rates are also 10 gallons per minute.

a) (8 points) Find the amount $x(t)$ of ethanol in tank 1.

b) (8 points) Find the amount $y(t)$ of ethanol in tank 2.

c) (6 points) Find the maximum amount of ethanol ever in tank 2.