Differential equations

Math 217 — Fall 2009

September Exam

This exam contains fourteen problems numbered 1 through 14. Problems 1 – 13 are multiple choice problems. Problem 14 is a free-response question.

Problem 1 \( \frac{dy}{dx} = 6x^2 + 3 \)

The general solution of the differential equation

\[ y' = 1 + x + y + xy \]

satisfies which of the following implicit equations.

A) \( \ln |1 + y| = x + C \) \quad B) \( \ln |1 + y| = x - x^2 + C \) \quad C) \( \ln |1 + y| = x + \frac{1}{2}x^2 + C \)

D) \( \ln |1 + y| = x - \frac{1}{2}x^2 + C \) \quad E) \( \ln |1 + y| = x + x^2 + C \) \quad F) \( \ln |1 + y| = \frac{1}{2}x + C \)

\[ y' = (1+x)(1+y) \]

\[ \int \frac{dy}{1+y} = \int (1+x)dx + C \]

\[ \ln |1+y| = x + \frac{1}{2}x^2 + C \]

C)
Problem 2

For which of the initial value problems

I) \( \frac{dy}{dx} + xy = x + \frac{1}{x}, \quad y(1) = 0, \)

II) \( \frac{dy}{dx} + xy = x + \frac{1}{x}, \quad y(0) = 1, \)

III) \( \frac{dy}{dx} = y^\frac{1}{3}, \quad y(0) = 0, \)

do the results from class guarantee the existence of a unique solution.

A) Only I  B) Only II  C) Only III  D) I and II  E) I and III  F) II and III

\[ A) \]

\[ II) \quad \frac{dy}{dx} = f(x, y) \]

\[ f(x, y) = x + \frac{1}{x} - xy \quad \text{not cont at \((0, 1)\)} \]

\[ III) \quad \frac{dy}{dx} = f(x, y) \]

\[ f(y) = y^\frac{1}{3} \quad \frac{\partial f}{\partial y} = \frac{1}{3} y^{-\frac{2}{3}} \quad \text{not cont at \((1, 0)\)} \]
Problem 3

Which of the following differential equations is exact?

A) \((3xy-x^3)dx+(3xy-3x^2y)\,dy = 0\)  
B) \(e^y \,dx + e^x \,dy = 0\)  
C) \(xy^2 - x^2y' = 0\)  
D) \(6xy^2 + (6x^2y+1)\frac{dy}{dx} = 0\)  
E) \(xy \,dx + xy \,dy = 0\)  
F) None of the above

D)
Problem 4

Suppose that in the absence of other factors a population of fish in a certain lake increases at a rate proportional to the size of the population. However, each day a fisherman catches 10 out of the lake. If \( P(t) \) is the number of fish after \( t \) days, which of the following differential equations models this situation.

A) \( \frac{dP}{dt} = -10k \) for some \( k > 0 \)  
B) \( \frac{dP}{dt} = 10 - kP \) for some \( k > 0 \)

C) \( \frac{dP}{dt} = 10kP \) for some \( k > 0 \)  
D) \( \frac{dP}{dt} = kP - 10 \) for some \( k > 0 \)

E) \( \frac{dP}{dt} = kP + 10 \) for some \( k > 0 \)  
F) None of the above

\( \boxed{D) \)
Problem 5

At noon a car starts from rest at point A and proceeds at constant acceleration along a straight road toward point B. If the car reaches B at 12:50 P.M. with a velocity of 60 mi/h, what is the distance from A to B?

A) 25 miles    B) 27 miles    C) 29 miles
D) 31 miles    E) 33 miles    F) 35 miles

\[ v = at \]

\[ t = \frac{50}{60} \text{ h} \]

\[ v = 60 \]

\[ 12 \cdot 60 = -a \frac{50}{60} \]

\[ a = -72 \]

\[ s = \frac{1}{2} at^2 = \frac{1}{2} (-72) \left( \frac{50}{60} \right)^2 = 22.5 \]
Problem 6

Solve

\[(x^2 + 1)y' + 3xy = 6x, \quad y(0) = 2.\]

What is \(y(4)\)? Pick the closest answer.

A) \(-2\)  B) \(-1\)  C) 0  D) 1  E) 2  F) 3

\[y = 2 \text{ is an equilibrium}\]

\[y(4) = 2 \quad E)\]
Problem 7

The solution to the initial value problem \( \frac{dy}{dx} + y = \sqrt{1 + x^4} \), \( y(1) = e \) is given by which of the following?

A) \( y = \frac{1}{x} \left( e + \int_1^x \sqrt{1 + t^4} \, dt \right) \)

B) \( y = \frac{1}{x} \left( 1 + \int_e^x \sqrt{1 + t^4} \, dt \right) \)

C) \( y = \frac{1}{x} \left( e + \sqrt{1 + x^4} \right) \)

D) \( y = \frac{1}{x} \left( e + \int_1^x t \sqrt{1 + t^4} \, dt \right) \)

E) \( y = \frac{1}{x} \left( 1 + \int_e^x \frac{\sqrt{1 + t^4}}{t} \, dt \right) \)

F) None of the above

First order linear equation

\((xy)' = \sqrt{1 + x^4}\)

\(xy = \int_1^x \sqrt{1 + t^4} \, dt + C\)

\(y(1) = e\)

\(e \cdot y(1) = \int_1^x \sqrt{1 + t^4} \, dt + C\)

\(C = e\)

\(xy = \int_1^x \sqrt{1 + t^4} \, dt + e\)

\(y = \frac{1}{x} \left( \int_1^x \sqrt{1 + t^4} \, dt + e \right)\)
Problem 8

Find the general solution to \( xy^2 \frac{dy}{dx} = y^3 + x^4 \). *Hint:* Use the substitution \( u = \frac{y}{x} \).

A) \( y = C(x^4 + y^4) \)  
B) \( y = \sqrt[3]{3x + C} \)  
C) \( y = \frac{1}{x} \sqrt[3]{3x + C} \)

D) \( y = (3x + C)^{-2/3} \)  
E) \( y = x \sqrt[3]{3x + C} \)  
F) None of the above

\[
\frac{dy}{dx} = \frac{y^3 + x^4}{y^2} = \frac{y}{x} + x - \frac{y}{u^2} \]

\[
\frac{dy}{dx} = \frac{y'}{x} = \frac{(u + xu^2) x - u x}{x^3} = \frac{u + x - u^2 - u}{x}
\]

\[
\frac{du}{dx} = u^{-2}
\]

\[
\int u^2 \, du = \int d\frac{1}{x}
\]

\[
\frac{1}{3} u^3 = x + C
\]

\[
u^3 = 3x + C \]

\[
u = \sqrt[3]{3x + C}
\]

\[
y = x \sqrt[3]{3x + C}
\]
Problem 9

Suppose a water tank is formed by revolving the curve $y = x^4$ around the $y$-axis. At time $t = 0$ a spherical hole is punched in the bottom of the tank. How large should the radius of this hole be in order for the water level to fall at a constant rate of 6 in/s? Pick the closest answer.

_Hint:_ Recall Torricelli's law: $A(y) \frac{dy}{dt} = -a \sqrt{2g\ y}$ where

- $y =$ height of the water level
- $A(y) =$ horizontal cross-sectional area of the tank at height $y$
- $a =$ area of the hole at the bottom of the tank
- $g =$ 32 ft/s²

A) 1 in  B) 2 in  C) 1/2 in  D) 4 in  E) 3 in  F) .87 in

$$A(y) = \pi x^2 = \pi \sqrt{\frac{y}{4}} \quad , \quad a = \pi r^2$$

$$-\frac{\pi \sqrt{\frac{y}{4}} \ dy}{dt} = -\pi r^2 \sqrt{2g\ y}$$

$$-6\ \text{in/s} = \frac{dy}{dt} = -r^2 \sqrt{2g} = -r^2 \sqrt{12.82} \quad \text{ft/s}$$

$$0.5 = r^2$$

$$\frac{1}{8} = \frac{1}{8}$$

$$2r \cdot \frac{1}{8} = r^2$$

$$r = \frac{1}{4}, \frac{5}{4}$$
Problem 10

Solve the initial value problem

\[ \frac{dy}{dx} = x\sqrt{x^2 + 9}, \quad y(-4) = 0. \]

Find \( y(0) \). Pick the closest answer.

A) \(-31.7\)  \hspace{0.5cm} B) \(31.7\)  \hspace{0.5cm} C) \(-32.7\)  \hspace{0.5cm} D) \(32.7\)  \hspace{0.5cm} E) \(-33.7\)  \hspace{0.5cm} F) \(33.7\)

\[
\begin{align*}
y' &= \int x\sqrt{x^2 + 9} \, dx \\
    &= \frac{1}{2} \int \sqrt{x^2 + 9} \, dx^2 \\
    &= \frac{1}{3} (x^2 + 9)^{3/2} + C \\
y(-4) &= C = \frac{1}{3} (16 + 9)^{3/2} + C \\
    &= \frac{1}{3} 25^{3/2} + C = \frac{1}{3} 125 + C \\
C &= -125/3 \\
y(0) &= \frac{1}{3} (0 + 9)^{3/2} - 125/3 \\
    &= (27 - 125/3)^{1/3} \\
    &= \left(\frac{98}{3}\right)^{1/3} \\
    &= -32.7 \\
\end{align*}
\]

\( \text{C) } \)
Problem 11

Which of the following equations has \( y = 1 \) as a stable critical point?

I) \( y' = y^2 - 1 \),

II) \( y' = y - 1 \),

III) \( y' = -y^2 + 1 \),

IV) \( y' = y^2 - 5y + 4 \).

A) I and II  B) I and III  C) I and IV  
D) II and III  E) II and IV  F) III and IV

\[ I \]
\[ \frac{\text{\underline{\text{\scriptsize 1}}}}{\text{\underline{\text{\scriptsize 1}}} - 1} + \quad \text{} + \quad y = 1 \text{ unstable} \]

\[ II \]
\[ \frac{\text{\underline{\text{\scriptsize 1}}}}{\text{\underline{\text{\scriptsize 1}}} - y} + \quad \text{} + \quad y = 1 \text{ unstable} \]

\[ III \]
\[ \frac{\text{\underline{\text{\scriptsize 1}}} y^2 - 1}{\text{\underline{\text{\scriptsize 1}}} y - 1} - \quad \text{} + \quad y = 1 \text{ stable} \]

\[ IV \]
\[ \frac{\text{\underline{\text{\scriptsize 1}}} y^2 - 5y + 4}{\text{\underline{\text{\scriptsize 1}}} y} + \quad \text{} + \quad y = 1 \text{ stable} \]
Problem 12

Which of the following is general solution (given implicitly) to the differential equation 
\((3x^2y - 3y) \, dx + (x^3 - 3x + 2y) \, dy = 0\)?

A) \(x^3 - xy^3 = C\)  B) \(x^3y - 3xy + y^2 = C\)  C) \(x^3 + xy^3 = C\)
D) \(3x^2 - 3 = C\)  E) \(x^3 + y^3 - 3xy = C\)  F) None of the above

\[
M = 3x^2y - 3y \\
N = x^3 - 3x + 2y
\]

\[
F_x = 3x^2y - 3y \\
F_y = x^3 - 3x + 2y
\]

\[
F = \int (3x^2y - 3y) \, dx + \int (x^3 - 3x + 2y) \, dy \\
F = \frac{1}{2}x^3y - 3xy + 2y \frac{1}{2}x^3 - 3x + 2y
\]

\[
F = \frac{1}{2}x^3y - 3xy + 2y
\]

\[
G = y^2 \\
G' = 2y
\]

\[
F = x^3y - 3xy + y^2 \\
x^3y - 3xy + y^2 = C
\]

\[\boxed{B}\]
Problem 13

Which of the following differential equations represents the slope field below?

A) $y' = x^2$  
B) $y' = y^2$  
C) $y' = x^2 - y^2$

D) $y' = x - y$  
E) $y' = y^2 - x^2$  
F) $y' = y - x$

$x = 0$

A) $y' = 0$, B) $y = y^2$ always positive

C) $y' = -y^2$ always negative

D) $y' = -y$

E) $y' = y^2$ always positive

F) $y' = y$

On y axis, slope fields is always negative

C)
Differential equations
Math 217 — Fall 2009

The following problem is a free-response question. You should justify your answers.

Problem 14 22''

A tank initially contains 50 gallons of pure water. Brine containing 1 lb. of salt per gallon of water enters the tank at a rate of 2 gallons per minute, while the (perfectly mixed) solution leaves the tank at the same rate.

a) Find the amount of salt $x(t)$ after $t$ minutes. 10 pts

At the time $t = T$ when the tank contains 10 lb. of salt, the drainage of the tank is partly clogged, so that the outflow rate is only 1 gallon per minute.

b) Find $T$. 4 pts

c) Find the amount of salt $x(t)$ for $t > T$. 8 pts

\[ Q' = Q_{in} - Q_{out} \]
\[ = 2 - \frac{Q}{50} \]
\[ Q' + \frac{Q}{25} = 2 \]
\[ e^{\frac{t}{25}} (Q' + \frac{Q}{25}) = 2 e^{\frac{t}{25}} \]
\[ (e^{\frac{t}{25}} Q)' = 2 e^{\frac{t}{25}} \]
\[ e^{\frac{t}{25}} Q = 50 e^{\frac{t}{25}} + C \]
\[ Q = 50 + C e^{\frac{t}{25}} \]

\[ Q(0) = 0 \]
\[ 0 = 50 + C \quad C = -50 \]
\[ Q = 50 - 50 e^{-\frac{t}{25}} \]

\[ Q(t) = 10 - 50 - 50 e^{-\frac{t}{25}} \]
\[ 40 = 50 e^{-\frac{t}{25}} \]
\[ 0.8 = e^{-\frac{t}{25}} \]
\[ \log_e 0.8 = -\frac{t}{25} \]
\[ -25 \log_e 0.8 \]

\[ T = -25 \log_e 0.8 \]

\[ Q'(t) = 2 - \frac{Q}{50+1} = 2 \]

\[ Q(t) = 2 + C e^{\frac{t}{25+1}} \]

\[ (50+t)Q = 100t + t^2 + C \]
\[ Q = \frac{100t + t^2 + C}{50+t} \]
\[ Q(0) = 10 \]
\[ Q(0) = \frac{C}{50} = 10 \quad C = 500 \]
\[ Q(t) = \frac{100t + t^2 + 500}{50 + t} \]

At time \( t \geq T \), there is
\[ Q(t-T) \text{ lb salt in the Tank} \]