1. Find the general solution of \( y' - \frac{y}{t} = t^3, \quad t > 0. \)

A) \( \frac{t^4}{3} + Ct \)

\( y' - \frac{y}{t} = t^3, \quad p = -\frac{1}{t} \)

I. F. \( \mu(t) = e^{\int p \, dt} = e^{\int \frac{-1}{t} \, dt} = e^{-\ln t} = \frac{1}{t} \)

B) \( 4t^3 + Ct \)

C) \( Ce^{\sin t} \)

D) \( \frac{e^{3t}}{7} + Ce^{-2t} \)

\( \mu \times y : \quad (\frac{y}{t})' = \frac{t^3}{t} = t^2 \)

E) \( \frac{e^{-5t}}{7} + Ce^{2t} \)

\( \int : \quad \frac{y}{t} = \frac{1}{8} t^3 + C \)

\( y(t) = \frac{t^4}{3} + Ct \)

F) \( Ce^{-3t} \)

G) \( Ce^{3t} \)

H) \( Ce^{\cos t} \)

I) \( C \sin t \)

J) None of the above
2. Find the general solution of \( \frac{dy}{dt} = -3y \).

A) \( \frac{t^4}{3} + Ct \)  
B) \( 4t^3 + Ct \)  
C) \( Ce^{3t} \)  
D) \( \frac{e^{3t}}{7} + Ce^{-2t} \)  
E) \( \frac{e^{-3t}}{7} + Ce^{2t} \)  
F) \( Ce^{-3t} \)  
G) \( Ce^{3t} \)  
H) \( Ce^{cost} \)  
I) \( Csint \)  
J) None of the above
3. Find the general solution of \( y' + 2y = e^{5t} \).

A) \( \frac{t^4}{3} + Ct \)

B) \( 4t^3 + Ct \)

C) \( Ce^{sin t} \)

D) \( \frac{e^{5t}}{7} + Ce^{-2t} \) \( \text{is linear} \)

\[ p = 2 \Rightarrow \mu(t) = e^{2t} \]

\[ \mu \times y : (y e^{2t})' = e^{5t} e^{2t} = e^{7t} \]

\[ \int: y e^{2t} = \frac{1}{7} e^{7t} + C \]

\[ y(t) = \frac{e^{5t}}{7} + Ce^{-2t} \]

E) \( \frac{e^{-5t}}{7} + Ce^{2t} \)

F) \( Ce^{-3t} \)

G) \( Ce^{3t} \)

H) \( Ce^{cos t} \)

I) \( C \sin t \)

J) None of the above
4. Let \( y(t) \) be the solution of the I.V.P. \( y' - y = y^4 \), \( y(0) = -1 \). Find \( y(3) \). (This Bernoulli equation I.V.P. has a particularly simple solution.)

\[
\begin{align*}
&\text{A) } -2 \\
&\text{B) } -1.5 \\
&\text{C) } -1 \\
&\text{D) } 1 \\
&\text{E) } 2.5 \\
&\text{F) } 3 \\
&\text{G) } 3.5 \\
&\text{H) } 4.5 \\
&\text{I) } 5 \\
&\text{J) None of the above}
\end{align*}
\]

\( y' - y = y^4 \) is Bernoulli.

\( \nu = y \), so use subs \( \nu = y^{-1} \). 

\[
\nu = y^{-1}, \quad \nu' = \frac{1}{y^3}, \quad y = \frac{1}{\nu^{1/3}}
\]

Then \( \nu' = -\frac{y^4}{3} \nu \).

\( \nu' + 3\nu = -3 \)

\( \text{W} e ^{3t}, \quad m = e^{3t} \)

\[
\begin{align*}
(\nu e^{3t})' &= -3e^{3t} \\
\nu e^{3t} &= -e^{3t} + C
\end{align*}
\]

\[
\nu(t) = \frac{C - e^{3t}}{e^{3t}}
\]

so \( y(t) = \frac{1}{\nu^{1/3}} = \left( \frac{1}{\nu} \right)^{1/3} \)

\[
\left( \frac{e^{3t}}{C - e^{3t}} \right)^{1/3}
\]

Use \( IC = -1 = y(0) = \left( \frac{1}{c-1} \right)^{1/3} \)

\[
\frac{1}{c-1} = -1 \Rightarrow c = 0
\]

\[
y(t) = \left( \frac{e^{3t}}{-e^{3t}} \right)^{1/3} = (-1)^{1/3}
\]

\( y(0) = -1 \)

So in particular \( y(3) = -1 \).
5. Let $y(t)$ be the solution to the I.V.P. $y' = \frac{t^2}{y^2}, \quad y(0) = 0$. Find $y(3)$.

A) -2
B) -1.5
C) -1
D) 1
E) 2.5
F) 3
g) 3.5
H) 4.5
I) 5
J) None of the above

\[ \frac{1}{3} y^3 = \frac{1}{3} t^3 + C \]
\[ y^3 = t^3 + C \]
\[ y(t) = \left( t^3 + C \right)^{1/3} \]

\[
\text{I.e.} \quad 0 = y(0) = \left( 0^3 + C \right)^{1/3} = C^{1/3} \quad \Rightarrow \quad C = 0
\]

\[ y(t) = (t^3)^{1/3} = t \]
\[ \therefore \quad y(3) = 3 \]
6. Let \( y(t) \) be the solution to the I.V.P. \( y' = 2t(y^2 + 1) \), \( y(0) = 0 \). Find \( y\left(\frac{\sqrt{\pi}}{2}\right) \).

A) -2  
B) -1.5  
C) -1  
D) 1  
E) 2.5  
F) 3  
G) 3.5  
H) 4.5  
I) 5  
J) None of the above
7. Find the general solution of \( y' - y \cos t = 0 \).

A) \( \frac{t^4}{3} + Ct \)

B) \( 4t^3 + Ct \)

C) \( Ce^{\sin t} \)

D) \( \frac{e^{3t}}{7} + Ce^{-2t} \)

E) \( \frac{e^{3t}}{7} + Ce^{2t} \)

F) \( Ce^{-3t} \)

G) \( Ce^{3t} \)

H) \( Ce^{\cos t} \)

I) \( C \sin t \)

J) None of the above

\[
\frac{dy}{dt} - y \cos t = 0 \quad \text{is solved:}
\]

\[
\frac{1}{y} \frac{dy}{dt} = \cos t \; dt
\]

\[
\ln|y| = \sin t + D
\]

\[
|y| = e^D e^{\sin t}
\]

\[
y = \pm e^D e^{\sin t}
\]

so \( y(t) = C e^{\sin t} \quad (C = \pm D) \)
8. A certain (hypothetical) radioactive element has a half-life of 20 hours. A sample of the substance is observed in the laboratory. At time \( t = 0 \), there are 100 g present. How many hours will it be until only 25 g remain?

A) 0
B) 5
C) 12
D) 17
E) 20
F) 23
G) 33
H) 40
I) 45
J) None of the above

**Method I:**

\[ 25 = \frac{1}{2} \cdot 100, \text{ so it will take} \]
\[ 20 + 20 = 40 \text{ hr.} \]

**Method II:**

\[ Q(t) = Q_0 e^{-rt} \] where \( r = \frac{\ln 2}{20} \) on.

\[ Q(t) = 100 e^{-0.034657t} \]

When \( t \) so

\[ 25 = 100 e^{-0.034657t} \]

\[ \Rightarrow t = \frac{\ln 0.0004}{-0.034657} \text{ hr.} \]
9. On a certain planet the acceleration due to gravity is \( g = 10 \text{ m/sec}^2 \) and the air causes a frictional drag force of \( \frac{1}{10} |\mathbf{v}| \text{ N} \) on a body traveling through it at \( \mathbf{v} \text{ m/sec} \). A cannon on the surface fires a 1 Kg projectile straight up at a velocity of 900 m/sec. To the nearest second, how long does it take the projectile to reach the top of its trajectory.

A) 0
B) 5
C) 12
D) 17
E) 20
F) 23
G) 33
H) 40
I) 45
J) None of the above

\[
\begin{align*}
\text{Forces:} & \\
\text{Grav:} & -10.1 \\
\text{Fr.} & -\frac{1}{10} \mathbf{v} \\
\text{Use} & \quad F = ma \\
\text{\quad} & \quad -\frac{1}{10} \mathbf{v} - 10 = 1. \mathbf{v} \\
& \quad \mathbf{v}^2 + \frac{1}{10} \mathbf{v} = -10 \\
& \text{is linear, general solution} \\
& \mathbf{v}(t) = -100 + C e^{-\frac{t}{10}} \\
\text{Use I.C.:} & \quad 900 = \mathbf{v}(0) = -100 + C \Rightarrow C = 1000 \\
\text{\quad} & \quad \mathbf{v}(t) = -100 + 1000 e^{-\frac{t}{10}} \\
\text{Note:} & \text{Prot return takes height h at time t,} \\
\text{when} & \quad \mathbf{v}(t_k) = 0 \\
\text{\quad} & \quad C = -100 + 1000 e^{-\frac{t_k}{10}} = 0 \\
\text{Solve:} & \quad t = 10 \times 10 = 28.026 \text{ secs}
\end{align*}
\]
10. Ceramic bricks are formed before firing in 30°C. They are then placed in an oven whose temperature is 1200°C after one hour in the oven, the temperature of the bricks is 500°C. The firing process will take them to attain a temperature of 1000°C. Which is the closest answer?

A) -2
B) -1.5
C) -1
D) 1
E) 2.5
F) 3
G) 3.5
H) 4.5
I) 5
J) None of the above

Also known:

500 = T(1) = \frac{dT}{dt}

\text{Solve for } t_k

So \ T(t) = 1200 - \text{new} \ t_k \text{ so } T(t+k)

1200 - 1170 = \text{Solve for } t_k
11. Certain microorganisms reproduce in such a way that the population \( P(t) \) at time \( t \) hours can be modeled using Verhulst's logistic equation with an intrinsic growth rate of \( r = .05 \) and an environmental carrying capacity (saturation level) of \( K = 10^6 \) individuals. A colony begins with 1000 individuals. At the time when the population reaches \( P = 10^7 \), how fast is the population changing, in millions of individuals per hour?

\[
\frac{dP}{dt} = r \left( 1 - \frac{P}{K} \right) P
\]

\[
\begin{align*}
A) & \quad -4.5 \\
B) & \quad -3.2 \\
C) & \quad -1 \\
D) & \quad -0.5 \\
E) & \quad 0 \\
F) & \quad 0.5 \\
G) & \quad 1 \\
H) & \quad 3.2 \\
I) & \quad 4.5 \\
J) & \quad None \ of \ the \ above
\end{align*}
\]
12. For what value of $a$ is $3x^3 +axy + 2 + (6y^2 - x^3 + y)\frac{dy}{dx} = 0$ an exact equation?

A) -2

B) -1.5

C) -1

D) 1

E) 2.5

F) 3

G) 3.5

H) 4.5

I) 5

J) None of the above
13. Which of the following is an implicit solution of the I.V.P.
\[ y^2 + (2xy + 1) \frac{dy}{dx} = 0, \quad y(1) = -2? \]

A) \( xy + y = 2 \)
B) \( x - xy = 2 \)
C) \( xy + x = 3 \)
D) \( x^2y + y^2 = 0 \)
E) \( xy^2 + y = 2 \)
F) \( x^2y + y = 2 \)
G) \( xy^2 - y = 1 \)
H) \( x^2y^2 + xy = 1 \)
I) \( x^3y - y^3 = 3 \)
J) None of the above

\[ M = y^2 \quad \Rightarrow \quad M_y = 2y \]
\[ N = 2xy + 1 \quad \Rightarrow \quad N_x = 2y \]
\[ \text{So Option C is exact} \]

\[ 1. \quad C'(y) = N - \sum M_y \, dx = N - \sum 2y \, dx \]
\[ = 2xy + 1 - 2xy = 1 \quad (\text{FCW of } y, \text{ no } x) \]
\[ 2. \quad C(y) = \sum dgy = y \]

\[ 2. \quad \psi(x,y) = \sum M \, dx + C \]
\[ = \sum y^2 \, dx + y = xy^2 + y \]

4. Implicit solution is
\[ xy^2 + y = C \]

Use I.E. which says "When \( x = 1 \), \( y = -2 \)"

\[ 1(-2)^2 + (-2) = C \quad \Rightarrow \quad C = 2 \]

I.e. is \( xy^2 + y = 2 \)
14. Let \( y(x) \) be the solution to the I.V.P. \( y' = x, \ y(0) = 1 \). Use Euler's method, with \( n = 2 \) intervals of subdivision, and \( h = 0.1 \) to find \( y_2 \), the Euler approximation to \( y(0.2) \).

   - **A)** \( 0 \)
     \[
     x_{k+1} = x_k + h = x_k + 0.1
     \]
   - **B)** \( 0.01 \)
     \[
     y_{k+1} = y_k + hf(x_k, y_k)
     = y_k + (0.1) x_k
     \]
   - **C)** \( 0.02 \)
     \[
     y_{k+1} = y_k + hf(x_k, y_k)
     \]
   - **D)** \( 0.1 \)
     \[
     y_{k+1} = y_k + hf(x_k, y_k)
     \]
   - **E)** \( 0.98 \)
     \[
     y_{k+1} = y_k + hf(x_k, y_k)
     \]
   - **F)** \( 0.99 \)
     \[
     y_{k+1} = y_k + hf(x_k, y_k)
     \]
   - **G)** \( 1.0 \)
     \[
     y_{k+1} = y_k + hf(x_k, y_k)
     \]
   - **H)** \( 1.01 \)
     \[
     y_{k+1} = y_k + hf(x_k, y_k)
     \]
   - **I)** \( 1.02 \)
     \[
     y_{k+1} = y_k + hf(x_k, y_k)
     \]
   - **J)** None of the above

\[\begin{align*}
  k=0 & \quad x_0 = 0, \quad y_0 = 1 \\
  k=1 & \quad x_1 = x_0 + 0.1 = 0 + 0.1 = 0.1 \\
 & \quad y_1 = y_0 + (0.1) x_0 = 1 + (0.1) 0 = 1 \\
  k=2 & \quad x_2 = x_1 + 0.1 = 0.1 + 0.1 = 0.2 \\
 & \quad y_2 = y_1 + (0.1) x_1 = 1 + (0.1) (0.1) = 1.01
\end{align*}\]
15. Find the exact solution $y(x)$ to the I.V.P. of problem 14, and then find the error committed in the approximation found there. i.e., find $|y_2 - y(0.2)|$.

A) 0
B) 0.01
C) 0.02
D) 0.1
E) 0.98
F) 0.99
G) 1.0
H) 1.01
I) 1.02
J) None of the above

**Exact Solution:**

1) $y' = x$
2) $y = \int x \, dx = \frac{1}{2} x^2 + c$

I.C. $1 = y(c) = \frac{1}{2} c^2 + c = c$

$y(x) = \frac{1}{2} x^2 + 1$

$\therefore y(x) = y(0.2) = \frac{1}{2} (0.2)^2 + 1 = 1.02$

So error

$= |y(0.2) - y_2| = |1.02 - 1.01| = 0.01$
A tank whose capacity is 100 liters initially contains 50 liters of water in which is dissolved 50 g of salt. Pure water enters the tank at \( r \) liters per minute, and after instant continuous mixing solution leaves the tank at one liter per min.

a) If \( r = 2 \), the tank will eventually overflow. After how many minutes does this occur?
b) What is the concentration of the solution in the tank at the moment of overflow?
c) Suppose that at the moment of overflow, \( r \) instantly changes to 1 liter per min. How many minutes after overflow does it take for the concentration to reach 0.1 g/l?

\[
\text{USE THE GENERAL EQN: } x' + \frac{V_{\text{out}}}{V(t)} x = \frac{V_{\text{in}}}{V(t)} C_{\text{in}}
\]

Here \( V_{\text{out}} = 1 \), \( C_{\text{in}} = 1 \), so:

\[
x' + \frac{1}{V(t)} x = 0
\]

a) If \( V_{\text{in}} = V = 2 \) and \( V_{\text{out}} = 1 \), \( V \) increases at 1 l/min, so \( V(t) = 50 + t \), overflow occurs when \( 100 = V(t) = 50 + t \) ⇒ \( t = 50 \) min

b) For \( V(t) = 50 + t \),

\[
x' + \frac{1}{50+t} x = 0
\]

Is linear.

Solve: \( x(t) = \frac{C}{50+t} \)

But know \( 50 = x(0) = \frac{C}{50} \) ⇒ \( C = 2500 \)

At overflow \( t = 50 \), \( x(50) = \frac{2500}{50+50} = 25 \) g

So concentration \( \text{conc}(50) = \frac{x(50)}{V(50)} = \frac{25}{100} = 0.25 \text{ g/l} \)

c) Now set \( t = 0 \) at overflow. Have \( x(0) = 25 \), \( V(t) = 100 \) so IVP: \( x' + \frac{1}{100} x = 0 \), \( x(0) = 25 \), solve IS:

\[
x(t) = 25 e^{-t/100}
\]

Therefore, concentration is then

\[
C(t) = \frac{x(t)}{V(t)} = \frac{1}{4} e^{-t/100}
\]

Need \( t \) so

\[
b(t) = c(t) = \frac{1}{4} e^{-t/100}
\]

Solve: \( C = -100 \ln(t) \)

\[
= 91.6291 \rightarrow \approx 91.6 \text{ min}
\]