Differential equations

Math 217 — Fall 2009

October Exam

This exam contains thirteen problems numbered 1 through 13. Problems 1 – 12 are multiple choice problems. Each problem counts 7 points. Problem 13 is a free-response question, which counts 16 points.

Problem 1

Compute the Wronskian of the functions $e^x \cos(x)$ and $e^x \sin(x)$.

A) 1  B) $e^x$  C) $e^{2x}$  D) $2e^x$  E) $e^x \cos(x) \sin(x)$  F) $2e^x \cos(x) \sin(x)$
Problem 2

Consider the functions \( y_1 = \cos^2(x), \ y_2 = \sin^2(x), \) and \( y_3 = \pi. \) Which of the following is true?

A) \( y_1, y_2 \) and \( y_3 \) are linearly independent on the real line.

B) \( y_1, y_2 \) and \( y_3 \) are linearly dependent on the real line.

C) \( y_1 \) and \( y_3 \) are linearly dependent on the real line.

D) \( y_2 \) and \( y_3 \) are linearly dependent on the real line.

E) There exist a constant \( k \) so that \( y_1(x) = ky_2(x) \) for all \( x. \)

F) The only constants \( c_1, c_2 \) and \( c_3 \) that satisfy \( c_1y_1 + c_2y_2 + c_3y_3 = 0 \) on the real line are \( c_1 = c_2 = c_3 = 0. \)
Problem 3

Suppose you have the following table of values for the two-variable function \( f(x, y) \).

<table>
<thead>
<tr>
<th>((x, y))</th>
<th>(f(x, y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,0)</td>
<td>-1</td>
</tr>
<tr>
<td>(0,1)</td>
<td>1</td>
</tr>
<tr>
<td>(1,1)</td>
<td>2</td>
</tr>
<tr>
<td>(1,2)</td>
<td>5</td>
</tr>
</tbody>
</table>

If \(y_n\) is the \(n\)-th approximation using the improved Euler method with step size \(h = 1\) of the initial value problem

\[
\frac{dy}{dx} = f(x, y), \quad y(0) = 1,
\]

what is \(y_1\)?

A) 2
B) 1
C) 4
D) 5
E) -1
F) 7
Problem 4

Consider the equation
\[ y^{(4)} + 2y' + 5y = g(t), \]
where \( g(t) \) is a non-zero function.

Suppose \( y_1, y_2 \) are solutions of the above nonhomogeneous differential equation, and \( y_3 \) is a solution of the corresponding homogeneous equation. Then only two of the following are solutions of the nonhomogeneous equation. Which two are they?

I) \( y_1 + y_2 \),  \hspace{1cm} II) \( 2y_1 - y_2 \),  \hspace{1cm} III) \( y_1 + y_3 \),  \hspace{1cm} IV) \( 2y_1 - y_3 \).

A) I and II \hspace{1cm} B) I and III \hspace{1cm} C) I and IV \hspace{1cm} D) II and III \hspace{1cm} E) II and IV \hspace{1cm} F) III and IV
Problem 5

Find the solution to the initial value problem

\[ y'' + y' = 0, \quad y(0) = 1, \quad y'(0) = 1. \]

A) 1 \quad B) e^{-x} \quad C) \frac{1}{2} + \frac{1}{2}e^{-x} \quad D) 2 - e^{-x} \quad E) 2e^{x} - 1 \quad F) \frac{3}{2} - \frac{1}{2}e^{x} \]
Problem 6

The function \( y = c_1 e^x + (c_2 + c_3 x) e^{-x} \) is a general solution of which of the following differential equations.

A) \( y^{(3)} - y = 0 \)
B) \( y'' + y''' - y - 1 = 0 \).
C) \( y^{(3)} - y'' + y' - y = 0 \).
D) \( y'' + 2y' + y = 0 \).
E) \( y^{(3)} + 2y'' + y' = 0 \).
F) \( y^{(3)} + y'' - y' - y = 0 \).
Problem 7

If \( y = Ax + B \) is a particular solution of \( y'' + y' + y = x - 1 \) what must \( A \) and \( B \) equal?

A) \( A = -1, \ B = 1 \)

B) \( A = 1, \ B = -1 \)

C) \( A = 1, \ B = 2 \).

D) \( A = -1, \ B = -2 \)

E) \( A = -2, \ B = 1 \).

F) \( A = 1, \ B = -2 \)
Problem 8

Find a particular solution of
\[ y'' + 4y = 2\cos(2x). \]

A) \( 2\cos(2x) \quad B) -\frac{1}{2}x\cos(2x) \quad C) \frac{1}{2}\cos(2x) \quad D) -\frac{1}{2}\sin(2x) \quad E) \frac{1}{2}x\sin(2x) \quad F) -2x\sin(2x) \]
Problem 9

Consider the differential equation

\[ y'' - 4y = e^x - x^2. \]

Using the method of undetermined coefficients, what is the correct form of a particular solution?

A) \( y_p = Ae^x + Bx^2 \)
B) \( y_p = Bxe^x + C + Dx + Ex^2 \)
C) \( y_p = Ae^x + B + Cx + Dx^2 \)
D) \( y_p = (A + Bx)e^x + Cx + Dx^2 \)
E) \( y_p = (A + Bx)e^x + Cx^2 \)
F) \( y_p = Ae^x + Bx \)
Problem 10

A motorboat weights 32,000 kg and its motor provides a thrust of 5,000 Newton. Assume that the water resistance is 100 Newton of each meter per second of the speed \( v \) of the boat. If the boat starts from rest, what is the maximum velocity that it can attain?

A) 1000 m/s  
B) 500 m/s  
C) 100 m/s  
D) 5,000 m/s  
E) 32,000 m/s  
F) 50 m/s
Problem 11

A body with mass $m = 1/2$ kg is attached to the end of spring stretched 2 meters by a force of 100 newtons (N). It is set in motion with initial position $x_0 = 1$ (m) and initial velocity $v_0 = -5$ (m/s). Find the amplitude of the position function of the body.

A) $\frac{1}{2}\sqrt{3}$  B) $\frac{1}{2}\sqrt{5}$  C) $\sqrt{3}$  D) $\sqrt{5}$  E) 1  F) 10
Problem 12

A spring-mass system is used as a scale as follows. The mass (whose weight one would like to know) is connected to a spring, the spring and mass is set in motion (we assume there is no friction or damping), and the period of the oscillations is observed.

During calibration it was found that a mass of 1 kg. corresponded to a period of 0.5 seconds. What is the mass of an object whose period is 1 second?

A) 0.25 kg.  B) 0.5 kg.  C) 1 kg.  D) \( \sqrt{2} \) kg.  E) 2 kg.  F) 4 kg.
The following problem is a free-response question. You should justify your answers. Please write your answers only on these two pages.

Problem 13

A) Find a general solution to the homogeneous differential equation

\[ y^{(3)} - y'' + y' - y = 0 \]

given that \( y = e^x \) is a solution.

B) Find a particular solution to

\[ y^{(3)} - y'' + y' - y = xe^x + 2\sin(x) \]

and use part A) to write the general solution.