Differential equations

Math 217 — Spring 2008

In-term exam March 4th.

This exam contains fourteen problems numbered 1 through 14. Problems 1 – 13 are multiple choice problems, which each count 6% of your total score. Problem 14 will be hand-graded and counts 22% of your total score.

Problem 1

Use the improved Euler method with step-size \( h = 1.0 \) to find an approximate value of \( y(2) \), where \( y(x) \) is the solution of

\[
\frac{dy}{dx} + \frac{3y^2}{x} = 0, \quad y(1) = 1.
\]

A) -7.00  B) -5.25  C) -3.50  D) -1.75  E) 0.00  F) 1.75
Problem 2

Find the solution to the differential equation

\[ y'' - 4y' + 5y = 0 \]

satisfying the initial conditions \( y(0) = 1, \ y'(0) = 5. \)

A) \( \cos x \)  B) \( \sin x \)  C) \( e^{2x} \cos x \)  D) \( 3e^{2x} \sin x \)  E) \( e^{2x}(\cos x + 3 \sin x) \)
F) \( e^{2x}(\cos x - 3 \sin x) \)
Problem 3

Consider the following pairs of functions.

i) \( f(x) = x, \quad g(x) = x^2 \),
ii) \( f(x) = e^x, \quad g(x) = x \),
iii) \( f(x) = 3x - 5, \quad g(x) = 9x - 12 \),
iv) \( f(x) = \cos^2 x, \quad g(x) = 1 - \sin^2 x \),
v) \( f(x) = e^x, \quad g(x) = e^{x+x} \),
vi) \( f(x) = x, \quad g(x) = x^{-1} \).

How many of the pairs i) - vi) are linearly independent on the interval \((0, 1)\)?

A) 1    B) 2    C) 3    D) 4    E) 5    F) 6
Problem 4

Find a particular solution of the nonhomogeneous differential equation

\[ y'' - 4y = e^{2x}. \]

A) 0  B) \(-\frac{1}{4}e^{2x}\)  C) \(\frac{1}{4}e^{2x}\)  D) \(-\frac{1}{4}xe^{2x}\)  E) \(\frac{1}{4}xe^{2x}\)  F) \(\frac{1}{4}x^2e^{2x}\)
Problem 5

Find the general solution of $y''' - 2y'' + y' = 0$.

A) $Ae^x + Be^z$  B) $Ae^x + Bxe^z$  C) $Ae^x + Be^z + C$  D) $Ae^x + Bxe^z + C$
E) $Ae^x + Be^z + Cx$  F) $Ae^x + Bxe^z + Cx$
Problem 6

A simple pendulum satisfies

\[ x'' + (2.17)^2 x = 0, \quad x(0) = 3, \quad x'(0) = 4. \]

What is the amplitude of its motion? Pick the closest value.

A) 3.5  B) 4.0  C) 4.5  D) 5.0  E) 5.5  F) 6.0
Problem 7

The general solution of \( y'' - 2y' + y = 0 \) is

\[ y_c(x) = (C_1 + C_2x)e^x. \]

Find the form of a particular solution of

\[ y'' - 2y' + y = x(1 - \cos x) + e^x. \]

A) \( Ax(1 - \cos x) + Be^x \)
B) \( (A_0 + A_1x)(1 - \cos x) + Be^x \)
C) \( (A_0 + A_1x) + (B_0 + B_1x) \cos x + Cx^2e^x \)
D) \( (A_0 + A_1x) + (B_0 + B_1x) \cos x + (C_0 + C_1x) \sin x + Dx^2e^x \)
E) \( (A_0 + A_1x) + (B_0 + B_1x)(\cos x + \sin x) + Cx^2e^x \)
F) \( (A_0 + A_1x) + (B_0 + B_1x) \cos x + (C_0 + C_1x) \sin x + (D_0 + D_1x + D_2x^2)e^x \)
Problem 8

A complementary solution of
\[ y'' + y = \frac{1}{\cos x} \]
is \( y_C = C_1 \cos x + C_2 \sin x \). Find a particular solution.

*Hint:* The formula
\[ \int \frac{\sin x}{\cos x} \, dx = -\ln |\cos x| + C \]
may be useful.

A) \( \frac{1}{\cos x} \)  B) \( x \sin x \)  C) \( \cos x \cdot \ln |\cos x| \)  D) \( x \sin x + \cos x \cdot \ln |\cos x| \)
E) \( x \sin x - \cos x \cdot \ln |\cos x| \)  F) \( \cos x + x \sin x \)
Problem 9

Consider the end-point problem

\[ y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) = 0. \]

For which of the following \( \lambda \) does there exist a non-trivial solution?

A) \( \lambda = -\pi^2 \)  B) \( \lambda = -1 \)  C) \( \lambda = 0 \)  D) \( \lambda = 1 \)  E) \( \lambda = \pi \)

F) \( \lambda = \pi^2 \)
Problem 10

The functions $y_1(x) = \cos x$, $y_2(x) = \sin x$, $y_3(x) = e^x$, and $y_4(x) = e^{-x}$ are four linearly independent solutions of $y^{(4)} - y = 0$. Which of the following functions are also solutions?

I) $\sin 2x = 2 \sin x \cos x$

II) $\cos x - \sin x$

III) $e^x \cos x + e^{-x} \cos x - \sin x$

A) Only I B) Only II C) Only III D) I and II E) I and III F) II and III
Problem 11

Apply Euler's method to approximate the solution of the initial value problem

\[ y' = x + \frac{1}{5} y, \quad y(0) = -3. \]

Find \( y(2) \) with step-size \( h = 1 \). Pick the closest value.

A) -3  B) -3.1  C) -3.2  D) -3.3  E) -3.4  F) -3.5
Problem 12

Which of the following differential equations describes a spring-mass system which exhibits the phenomenon of resonance?

A) \( x'' + 3x' + 2x = 0 \)
B) \( x'' + 3x' + 2x = \cos 2t \)
C) \( x'' + 3x' + 2x = \sin t \)
D) \( x'' + x = \frac{1}{2} \sin t \)
E) \( x'' + x = 0 \)
F) \( x'' + x = \cos 2t \)
Problem 13

Write down the characteristic equation of \( y'' - 7y' + 6y = 2x \).

A) \( r^3 - 7r + 6 = 0 \)   B) \( r^3 - 7r + 6 = 2 \)   C) \( r^3 - 7r = 6 \)   D) \( r^4 - 7r^2 + 6r = 0 \)
E) \( r^4 - 7r^2 + 6r = 2 \)   F) \( r - 7r + 6r = 0 \)
The following problem will be hand-graded. To earn full credit you need to justify your answers.

Problem 14

a) (8 points) Show that $y_1(x) = e^{-3x} \cos 4x$ and $y_2(x) = e^{-3x} \sin 4x$ are solutions of

$$y'' + 6y' + 25y = 0,$$

and compute their Wronskian.

b) (6 points) Find a particular solution $y_p(x)$ of

$$y'' + 6y' + 25y = 195 \cos 2x.$$

c) (8 points) Consider a damped mass-and-spring system with mass $m = 2$, damping constant $c = 12$, and spring constant $k = 50$ under the influence of the external force $F_E(t) = 390 \cos 2t$. Assume the system starts with $x(0) = 5$ and $x'(0) = 2$.

Set up a differential equation describing the motion $x(t)$ of the mass, and solve it.