Name:
ID:
Section:

This exam has 14 questions:

• 12 multiple choice worth 6 points each.
• 1 hand graded worth 28 points.

Important:

• No graphing calculators!
• For the multiple choice questions, mark your answer on the answer card.
• Show all your work for the written problems. You will be graded on the ease of reading your solution.
• You are allowed a $3 \times 5$ note card for the exam.

1. Multiply the matrices

\[
\begin{pmatrix}
1 & 2 & -2 \\
-3 & 2 & 0 \\
5 & -2 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 & -2 \\
-1 & 2 & 1 \\
4 & 4 & 0
\end{pmatrix}
\]

Add up all the entries in your answer and select the closest answer below.

(a) $-28$
(b) $-21$
(c) $-14$
(d) $-7 \rightarrow \text{CORRECT}$
(e) 0
(f) 7
(g) 14
(h) 21
(i) 28
(j) 34
(k) 43

Solution:

\[
\begin{pmatrix}
-10 & -3 & 0 \\
-2 & 1 & 8 \\
6 & 5 & -12
\end{pmatrix} \rightarrow -7
\]
2. Consider the system

\[
X' = \begin{bmatrix}
3 & -2 & 0 \\
-1 & 3 & -2 \\
0 & -1 & 3
\end{bmatrix} X
\]

This system has solutions below

\[
X_1(t) = e^{2t} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \quad X_2(t) = e^{4t} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \quad X_3(t) = e^{6t} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}
\]

Let \(W(t) = W(X_1, X_2, X_3)\) be the Wronskian. Find \(W(0)\). Choose the closest answer.

(a) \(-20\)
(b) \(-16\) → CORRECT \(W = -16e^{12t}\)
(c) \(-12\)
(d) \(-8\)
(e) \(-4\)
(f) 0
(g) 4
(h) 8
(i) 12
(j) 16
(k) 20

**Solution:**

\[
W = \begin{vmatrix}
2e^{2t} & 2e^{4t} & 2e^{6t} \\
2e^{2t} & 0 & -2e^{6t} \\
e^{2t} & -1e^{4t} & 1e^{6t}
\end{vmatrix} = -16e^{12t}
\]
3. Consider the initial value problem

\[ x' = x^2 - 2y - t \quad x(0) = 1 \]
\[ y' = 5x - 4y + t \quad y(0) = -2 \]

Using Runge-Kutta with step size \( h = 0.5 \) to approximate \( y(0.5) \). Choose the closest answer.

(a) -2.0
(b) -1.5
(c) -1.0
(d) -0.5
(e) 0
(f) 0.5
(g) 1.0
(h) 1.5
(i) 2.0 → CORRECT 1.914683
(j) 2.5
(k) 3.0

Solution:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( t_n )</th>
<th>( X_n )</th>
<th>( K_1 )</th>
<th>( K_2 )</th>
<th>( K_3 )</th>
<th>( K_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>1</td>
<td>5</td>
<td>2.3125</td>
<td>2.9905</td>
<td>0.085594</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-2</td>
<td>13</td>
<td>6.5</td>
<td>9.6406</td>
<td>1.6949</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>2.3076</td>
<td>1.91468</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
4. Convert the initial value problem to a first order system and use Euler’s method with step size \( h = 0.5 \) to approximate \( x(1.0) \)

\[
x'' - 2(x')^2 + x^2 t = -1 \quad x(0) = 1, x'(0) = -1
\]

Choose the closest answer.

(a) 4.40 \( \rightarrow \) CORRECT
(b) 4.45
(c) 4.50
(d) 4.55
(e) 4.60
(f) 4.65
(g) 4.70
(h) 4.75
(i) 4.80
(j) 4.85
(k) 4.90

**Solution:**

Equation transforms to

\[
x'_1 = x_2 \\
x'_2 = 2x_2^2 - x_1^2 t - 1 \\
x_1(0) = 1 \\
x_2(0) = -1
\]

<table>
<thead>
<tr>
<th>( n )</th>
<th>( t_n )</th>
<th>( X_n )</th>
<th>( K = F(t, X_n) )</th>
</tr>
</thead>
</table>
| 0      | 0.0    | \[
|        |        | \frac{1}{1} | \frac{-1}{1} |
| 1      | 0.5    | \[
|        |        | \frac{0.5}{-0.5} | \frac{-0.5}{-0.625} |
| 2      | 1.0    | \[
|        |        | \frac{0.25}{-0.8125} |

So, \( x(0) \approx 0.25 \).
5. Which of the following triplets of functions are linearly independent?

I. \( f(x) = e^x, \ g(x) = e^{-x}, \ h(x) = 1 \)
II. \( f(x) = e^x, \ g(x) = \cos x, \ h(x) = \sin x \)
III. \( f(x) = x, \ g(x) = x^2, \ h(x) = x^2 - x \)
IV. \( f(x) = \cos^2 x, \ g(x) = \sin^2 x, \ h(x) = 1 \)

(a) II only
(b) IV only
(c) I and II only → CORRECT
(d) I and IV only
(e) II and III only
(f) I, II and III only
(g) I, II and IV only
(h) II, III and IV only
(i) I, II, III and IV

Solution: We can compute Wronskians to answer this question (you can also see some of them are linearly independent by seeing that one of the functions is a combination of the others).

I. \( W(f, g, h) = 2 \neq 0 \)
II. \( W(f, g, h) = 2e^x \neq 0 \)
III. \( W(f, g, h) = 0, \) also note \( h = f - g. \)
IV. \( W(f, g, h) = 0, \) also note \( h = f + g. \)
6. Consider a mass-spring with mass \( m = 1 \), \( c = 0 \) (no damping) and \( k = 25 \). An input force satisfies \( F(t) = F_0 \sin \omega t \). For what value of \( \omega \), if any, is there resonance? Choose the closest answer.

(a) \( \omega = 0 \)
(b) \( \omega = 1 \)
(c) \( \omega = 4 \)
(d) \( \omega = 5 \) \( \rightarrow \) **CORRECT**
(e) \( \omega = 8 \)
(f) \( \omega = 9 \)
(g) \( \omega = 16 \)
(h) \( \omega = 25 \)
(i) No resonance possible
(j) None of the above

**Solution:** Resonance occurs when \( \omega = \omega_0 = \sqrt{\frac{k}{m}} = 5 \).

7. Choose the correct form of a particular solution to the differential equation below using the method of undetermined coefficients.

\[ y^{(3)} - y'' - 4y' = \sin x + x^2 - 5 \]

(a) \( y_p = A \sin x + B + Cx^2 \)
(b) \( y_p = A \cos x + B + Cx^2 \)
(c) \( y_p = A \sin x + B + Cx + Dx^2 \)
(d) \( y_p = A \cos x + B + Cx + Dx^2 \)
(e) \( y_p = A \cos x + B \sin x + C + Dx + Ex^2 \)
(f) \( y_p = A \cos x + B \sin x + x(C + Dx + Ex^2) \) \( \rightarrow \) **CORRECT**
(g) \( y_p = x(A \cos x + B \sin x) + C + Dx + Ex^2 \)
(h) \( y_p = x(A \cos x + B \sin x) + x(C + Dx + Ex^2) \)
(i) \( y_p = x(A \cos x + B \sin x) + C + Dx + Ex^2 + Fx^3 \)
(j) \( y_p = x(A \cos x + B \sin x) + x(C + Dx + Ex^2 + Fx^3) \)
(k) The method of undetermined coefficients can not be used with this differential equation or some other answer.

**Solution:** It is important to note that the solution to the homogeneous equation is

\[ y_c = c_1 + e^{x/2} \left[ c_2 \cos \left( \frac{t\sqrt{17}}{2} \right) + c_3 \sin \left( \frac{t\sqrt{17}}{2} \right) \right] \]

Actually, all you have notice is that \( y = c_1 \) is part of \( y_c \) and \( y = \sin x \) is not part of \( y_c \). Once you do this, you should be able to see the correct form.
8. Select the differential equation that has \( y = 4e^{2x} - e^{-2x} \cos 2x \) as a solution.

(a) \( y^{(4)} - 6y''' + 15x'' - 17y' + 10 = 0 \)
(b) \( y''' + 2y'' - 16y = 0 \quad \longrightarrow \text{CORRECT} \)
(c) \( y''' + 2y'' - 3y' - 10y = 0 \)
(d) \( y''' + y'' - 4y' - 4y = 0 \)
(e) \( y''' - y'' - 4y' + 4y = 0 \)
(f) \( y''' - 3y' + 2y = 0 \)
(g) \( y''' + 2y'' - y' - 2y = 0 \)
(h) \( y''' + 4y'' + 5y' + 2 = 0 \)
(i) All of the above have the given solution
(j) None of the above have the given solution

**Solution:** If this is a solution then the roots of the characteristic polynomial must be 2 and \(-2 \pm 2i\). Therefore the characteristic polynomial must have, as a factor:

\[
(r - 2)[r - (-2 + 2i)][r - (-2 - 2i)] = r^3 + 2r^2 - 16
\]
9. To solve non-homogeneous equations we discussed the method of undetermined coefficients and variation of parameters. Which of the following equations does the method of undetermined coefficients apply?

- I. $3y''' - 2y'' + 1y' = e^{2x} + \cos(x^2)$
- II. $xy''' - x^3y'' + x^2y' = \cos x + \sin x$
- III. $2y''' + 5y'' - \pi y' = x \cos x + \sin x$
- IV. $4y''' + 7y' - 13y = xe^{x^2}$

(a) I only
(b) II only
(c) III only $\rightarrow$ CORRECT
(d) IV only
(e) I and II only
(f) I and III only
(g) I and IV only
(h) II and III only
(i) II and IV only
(j) III and IV only
(k) Some other combination or none of the above

**Solution:** I. and IV. can not be solved using undetermined coefficient because the derivatives of the left hand side are not spanned by a finite set of linearly independent functions.

II. is ruled out because of the $x^2$ in the differential equation—you need constant coefficients to use the method of undetermined coefficients.
10. Solve the initial value problem

\[ y'' - 4y' + 4y = e^{2x}; \quad y(0) = 0, \quad y'(0) = 0. \]

What is \( y(2) \)?

(a) 0  
(b) 1  
(c) 2  
(d) \( \frac{1}{2}e^2 \)  
(e) \( 2e^2 \)  
(f) \( \frac{1}{4}e^4 \)  
(g) \( \frac{1}{2}e^4 \)  
(h) \( e^4 \)  
(i) \( 2e^4 \) \( \rightarrow \) **CORRECT**  
(j) \( 4e^4 \)

**Solution:** Factor the characteristic equation and find complementary solution \( y_c = c_1e^{2x} + c_2xe^{2x} \). Use the method of undetermined coefficients to find \( y_p \). The correct form for \( y_p \) should be \( y_p = x^2(Ae^{2x}) \). Plugging in and solving gives \( A = \frac{1}{2} \) which gives a general solution

\[ y = c_1e^{2x} + c_2xe^{2x} + \frac{x^2}{2}e^{2x} \]

Using the initial conditions to solve for \( c_1 \) and \( c_2 \) gives \( c_1 = c_2 = 0 \) and therefore

\[ y = \frac{x^2}{2}e^{2x} \]

And therefore \( y(2) = 2e^4 \).
11. A mass-spring system with forced oscillation satisfies the following initial value problem:

\[ x'' + 4x = \sin 2t; \quad x(0) = 0, \quad x'(0) = 0 \]

Find \( x(\pi) \).

(a) \(-2\pi\)
(b) \(-\pi/2\)
(c) \(-\pi/2 - 1/2\)
(d) \(-\pi/4\) \(\rightarrow\) CORRECT
(e) 0
(f) \(\sqrt{2}/2\)
(g) \(\pi/4\)
(h) \(\pi/2 + 1/2\)
(i) \(\pi/2 - 1/2\)
(j) \(2\pi\)

**Solution:** First find \( x_c = c_1 \cos 2t + c_2 \sin 2t \). Then, use the method of undetermined coefficients to find a particular solution. But, be careful because you have to use \( x_p = t(A \cos 2t + B \sin 2t) \). Plugging this in should give you \( A = -\frac{1}{4} \) and \( B = 0 \). Therefore the general solution is

\[ x = c_1 \cos 2t + c_2 \sin 2t - \frac{t}{4} \cos 2t \]

Now, using the initial condition gives \( c_1 = 0 \) and \( c_2 = \frac{1}{8} \). Therefore the solution we seek is

\[ x = \frac{1}{8} \sin 2t - \frac{t}{4} \cos 2t \]

and \( x(\pi) = -\frac{\pi}{4} \)
Consider the eigenvalue problem
\[ y'' - 2y' + \lambda y = 0; \quad y(0) = 0, \quad y(1) = 0 \]

Which of the following is an eigenvalue?

I. 1
II. \(2\pi\)
III. \(\pi^2 + 1\)
IV. \(4\pi^2 + 1\)

(a) I only
(b) II only
(c) III only
(d) IV only
(e) I and II only
(f) I and III only
(g) I and IV only
(h) II and III only
(i) II and IV only

(j) III and IV only \(\rightarrow \) CORRECT
(k) Some other combination

**Solution:** The characteristic equation for this differential equation is \(r^2 - 2r + \lambda = 0\). Solving gives \(r = 1 \pm \sqrt{1 - \lambda}\).

If \(1 - \lambda > 0\) then the general solution is
\[ y = c_1 e^{r_1 x} + c_2 e^{r_2 x} \]

where \(r_1, r_2\) are the real roots. Plugging in the boundary values gives the equations
\[ c_1 + c_2 = 0 \]
\[ c_1 e^{r_1} + c_2 e^{r_2} = 0 \]

which has for a solution \(c_1 = c_2 = 0\), so this does not give eigenvalues.

If \(1 - \lambda = 0\) then the general solution is
\[ y = c_1 e^x + c_2 xe^x \]

Plugging in the boundary values gives the equations
\[ c_1 = 0 \]
\[ c_1 e + c_2 e = 0 \]

which gives \(c_1 = c_2 = 0\), so this does not give eigenvalues.
If $1 - \lambda < 0$ then the general solution is

$$y = e^x \left[ c_1 \cos \left( x \sqrt{\lambda - 1} \right) + c_2 \sin \left( x \sqrt{\lambda - 1} \right) \right]$$

Plugging in the boundary values gives the equations

$$c_1 = 0$$

$$c_1 e \cos \left( \sqrt{\lambda - 1} \right) + c_2 e \sin \left( \sqrt{\lambda - 1} \right) = 0$$

which gives that for $\lambda$ to be an eigenvalue we must have $\sin \sqrt{\lambda - 1} = 0$, which means $\sqrt{\lambda - 1} = n\pi$ and $\lambda = n^2 \pi^2 + 1$. 
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Note: You will be graded on the readability of your work. Use the back of this sheet, if necessary.

13. (a) Find all eigenvalues and eigenvectors of the matrix

\[
A = \begin{bmatrix}
2 & 1 & 0 \\
0 & 1 & 2 \\
2 & 2 & 2 \\
\end{bmatrix}
\]

(b) Find the general solution to the system

\[
X' = AX
\]

where \( A \) is defined above and \( X = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \)

Solution:

(a) Solve \( \det(A - \lambda I) = 0 \):

\[
\begin{vmatrix}
2 - \lambda & 1 & 0 \\
0 & 1 - \lambda & 2 \\
2 & 2 & 2 - \lambda \\
\end{vmatrix} = -\lambda^3 + 5\lambda^2 - 4\lambda = -\lambda(\lambda - 4)(\lambda - 1)
\]

Thus the eigenvalues are \( \lambda_1 = 0, \lambda_2 = 1 \) and \( \lambda_3 = 4 \).

To find the eigenvectors we solve.

\( \lambda = 0 \): Let \( \mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \). We need to solve \((A - 0 \cdot I)\mathbf{v} = \mathbf{0}\) which gives the system

\[
\begin{bmatrix}
2 & 1 & 0 \\
0 & 1 & 2 \\
2 & 2 & 2 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 2 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

This gives equations \( a - c = 0 \) and \( b + 2c = 0 \). Therefore, if we let \( c = 1 \) then we get the eigenvector \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \).

\( \lambda = 1 \): Let \( \mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \). We need to solve \((A - 1 \cdot I)\mathbf{v} = \mathbf{0}\) which gives the system

\[
\begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 2 \\
2 & 2 & 1 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

This gives equations \( c = 0 \) and \( a + b = 0 \). Therefore, if we let \( b = 1 \) then we get the eigenvector \( \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \).
$\lambda = 4$: Let $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. We need to solve $(A - 4 \cdot I)v = 0$ which gives the system

$$
\begin{bmatrix}
-2 & 1 & 0 & 0 \\
0 & -3 & 2 & 0 \\
2 & 2 & -2 & 0
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 0 & -\frac{1}{3} & 0 \\
0 & 1 & -\frac{2}{3} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

This gives equations $a - \frac{1}{3}c = 0$ and $b - \frac{2}{3}c = 0$. Therefore, if we let $c = 3$ then we get the eigenvector $v_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(b) Each eigenvalue-eigenvector pair gives a solution. The general solution is

$$X = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + c_3 e^{\lambda_3 t} v_3$$

$$= c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$