# Name:  
# ID:  
# Section:

This exam has 16 multiple choice questions.

Important:
- No graphing calculators!
- For the multiple choice questions, mark your answer on the answer card.
- Show all your work for the written problems. You will be graded on the ease of reading your solution.
- You are allowed a 3 × 5 note card for the exam.

<table>
<thead>
<tr>
<th>Function</th>
<th>Transform</th>
<th>Function</th>
<th>Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>f′(t)</td>
<td>sF(s) − f(0)</td>
<td>1</td>
<td>1/s</td>
</tr>
<tr>
<td>f″(t)</td>
<td>s²F(s) − sf(0) − f′(0)</td>
<td>t</td>
<td>1/s²</td>
</tr>
<tr>
<td>∫₀ᵗ f(τ) dτ</td>
<td>F(s)</td>
<td>tⁿ</td>
<td>n!/sⁿ⁺¹</td>
</tr>
<tr>
<td>eᵃᵗf(t)</td>
<td>F(s − a)</td>
<td>tᵃ</td>
<td>(a + 1)/sᵃ⁺¹</td>
</tr>
<tr>
<td>u(t − a)f(t − a)</td>
<td>e⁻ᵃˢF(s)</td>
<td>1/√πt</td>
<td>s/√s</td>
</tr>
<tr>
<td>(f ∗ g)(t) = ∫₀ᵗ f(τ)g(t − τ) dτ</td>
<td>F(s)G(s)</td>
<td>cos kt</td>
<td>s/s² + k²</td>
</tr>
<tr>
<td>tf(t)</td>
<td>−F′(s)</td>
<td>sin kt</td>
<td>s² + k²</td>
</tr>
<tr>
<td>1/t f(t)</td>
<td>∫₀⁺∞ F(σ) dσ</td>
<td>sinh kt</td>
<td>k/(s² − k²)</td>
</tr>
<tr>
<td>f(t), period p</td>
<td>1/1 − e⁻ᵃˢ ∫₀⁺ᵖ e⁻ˢᵗf(t) dt</td>
<td>sinh kt</td>
<td>1/(s² + k²)²</td>
</tr>
<tr>
<td>u(t − a)</td>
<td>e⁻ᵃˢ/s</td>
<td>1/(2k³)(sin kt − kt cos kt)</td>
<td>1/(s² + k²)²</td>
</tr>
<tr>
<td>eᵃᵗ</td>
<td>1/s − a</td>
<td>k/2k sin kt</td>
<td>(s² + k²)²</td>
</tr>
<tr>
<td>tⁿeᵃᵗ</td>
<td>n!/sⁿ⁺¹</td>
<td>1/(2k sin kt + kt cos kt)</td>
<td>s²/(s² + k²)²</td>
</tr>
</tbody>
</table>

\[
\sin(A + B) = \sin A \cos B + \cos A \sin B \\
\cos(A + B) = \cos A \cos B - \sin A \sin B \\
2 \cos A \cos B = \cos(A - B) + \cos(A + B) \\
2 \sin A \sin B = \cos(A - B) - \cos(A + B) \\
2 \sin A \cos B = \sin(A - B) + \sin(A + B)
\]
1. Solve the differential equation

\[ \frac{dy}{dx} = 3x^2y - y, \quad y(1) = 2 \]

Find \( y(2) \) and select the closest answer.

(a) 0
(b) 10
(c) 25
(d) 50
(e) 100
(f) 200
(g) 500
(h) 800 \quad \rightarrow \text{CORRECT}
(i) 1000
(j) 1500
(k) 2500

\textbf{Solution:} This equation is separable and has solution

\[ y = Ce^{x^3-x} \]
\[ y = 2e^{x^3-x} \]
\[ y(2) \approx 806.86 \]
2. Solve the initial value problem

\[ y' + \frac{1}{x}y = 2 \sin(x^2), \quad y(\sqrt{\pi}) = 0 \]

What is \( y(2.5) \)?

(a) \(-1.0\)
(b) \(-0.9\)
(c) \(-0.8 \quad \textbf{CORRECT} \quad -0.79977979\)
(d) \(-0.7\)
(e) \(-0.6\)
(f) \(-0.5\)
(g) \(-0.4\)
(h) \(-0.3\)
(i) \(-0.2\)
(j) \(-0.1\)
(k) \(0.0\)

**Solution:** First order linear equation.

\[
y = \frac{C - \cos x^2}{x}
\]

\[
y = \frac{-1 - \cos x^2}{x}
\]
3. Solve the initial value problem

\[ xy^2 y' = x^3 + y^3, \quad y(1) = 0 \]

What is \( y(2) \)?

(a) \(-5\)
(b) \(-2.5\)
(c) 0
(d) 2.5 \(\rightarrow \text{CORRECT} \ 2.53\)
(e) 5
(f) 7.5
(g) 10
(h) 12.5
(i) 15
(j) 17.5
(k) 20

**Solution:** Homogeneous, let \( v = \frac{y}{x} \):

\[ xv' + v = \frac{1}{v^2} + v \]
\[ v = \sqrt[3]{3 \ln x} \]
\[ y = x^{\sqrt[3]{3 \ln x}} \]
4. Solve the initial value problem

\[ ye^{xy} + (xe^{xy} + 1)y' = 0, \quad y(0) = 1 \]

What is \( x \) when \( y = 0.5? \)

(a) 0.6  
(b) 0.7  
(c) 0.8 \( \rightarrow \) CORRECT 0.8109  
(d) 0.9  
(e) 1.0  
(f) 1.1  
(g) 1.2  
(h) 1.3  
(i) 1.4  
(j) 1.5  
(k) 1.6

Solution: Exact

\[ e^{xy} + y = C \]
\[ e^{xy} + y = 2 \]
\[ e^{x/2} + \frac{1}{2} = 2 \implies x = 2 \ln(3/2) \]
5. Find the equilibrium solutions of the autonomous equation

\[ y' = y^3 + y^2 - 2y \]

and determine which equilibria are stable.

(a) Stable: \( y = -2, \ y = 1 \); Unstable: \( y = 0 \)
(b) Stable: \( y = -2, \ y = 2 \); Unstable: \( y = 0 \)
(c) Stable: \( y = -1, \ y = 1 \); Unstable: \( y = 0 \)
(d) Stable: \( y = -1, \ y = 2 \); Unstable: \( y = 0 \)
(e) Stable: \( y = 0, \ y = 3 \); Unstable: \( y = 1 \)
(f) Stable: \( y = 1, \ y = 3 \); Unstable: \( y = 2 \)
(g) Stable: \( y = 0 \); Unstable: \( y = -2, \ y = 1 \) \( \longrightarrow \) CORRECT
(h) Stable: \( y = 0 \); Unstable: \( y = -2, \ y = 2 \)
(i) Stable: \( y = 0 \); Unstable: \( y = -1, \ y = 1 \)
(j) Stable: \( y = 1 \); Unstable: \( y = 0, \ y = 3 \)
(k) Stable: \( y = 2 \); Unstable: \( y = 1, \ y = 3 \)

Solution:

\[ y = y(y - 1)(y + 2) \]

Equilibrium solutions are \( y = -2, \ y = 0, \ y = 1 \). Only \( y = 0 \) is stable.
6. Use Runge-Kutta with step size $h = 0.5$ to estimate $y(1)$.

\[ y' = x^2 + xy^2, \quad y(0) = 1 \]

(a) 1.9
(b) 2.0
(c) 2.1
(d) 2.2
(e) 2.3
(f) 2.4
(g) 2.5
(h) 2.6
(i) 2.7 \rightarrow \text{CORRECT}
(j) 2.8
(k) 2.9

Solution:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x_n$</th>
<th>$y_n$</th>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$k_3$</th>
<th>$k_4$</th>
<th>$k$</th>
</tr>
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<tbody>
<tr>
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<td>0.0</td>
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<td>0.3125</td>
<td>0.3531</td>
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<td>2</td>
<td>1.0</td>
<td>2.7127</td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>
7. Solve the initial value problem

\[ y'' - 2y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 1 \]

What is \( y(1) \)?

(a) 1.1
(b) 1.6
(c) 1.9
(d) 2.3 \( \rightarrow \) **CORRECT** \( e \sin 1 \approx 2.28735529 \)
(e) 2.5
(f) 2.7
(g) 2.9
(h) 3.1
(i) 3.5

**Solution:**

\[
\begin{align*}
    y &= e^x (c_1 \cos x + c_2 \sin x) \\
    y &= e^x \sin x
\end{align*}
\]
8. Which of the following are solutions to the differential equation

\[ y'' + 2y' - 3y = 2 \cos x + 6 \sin x \]

I. \( y = -\sin x - \cos x \)
II. \( y = e^x - \sin x - \cos x \)
III. \( y = e^{-3x} + e^x \)
IV. \( y = e^{-3x} + e^x - \sin x \)

(a) I only
(b) II only
(c) IV only
(d) I and II only \( \rightarrow \text{CORRECT} \)
(e) I and IV only
(f) II and III only
(g) I, II and III only
(h) I, II and IV only
(i) II, III and IV only
(j) I, II, III and IV
(k) Some other answer

Solution: General solution is

\[ y = -\cos x - \sin x + c_1 e^x + c_2 e^{-3x} \]
9. For what value of $\alpha$ will the following mass-spring system have resonance?

$$\alpha^2 x'' + x = \sin 2t$$

(a) 0 
(b) 1/8 
(c) 1/4 
(d) 1/2 \quad \text{CORRECT} 
(e) 1/\sqrt{2} 
(f) \sqrt{2} 
(g) 2 
(h) 4 
(i) 8 
(j) No value of $\alpha$ will cause resonance.

**Solution:** The solution to the homogeneous problem is

$$x = c_1 \cos(t/\alpha) + c_2 \sin(t/\alpha)$$

Therefore resonance will occur when $\frac{1}{\alpha} = 2$, or when $\alpha = \frac{1}{2}$. 
10. Consider the differential equation modelling a mass-spring system.

\[ 31x'' + 16x' + 2x = 0 \]

(a) This equation is an example of resonance
(b) This equation is an example of overdamping → **CORRECT**
(c) This equation is an example of underdamping
(d) This equation is an example of critical damping
(e) None of the above

**Solution:** The equation \( mx'' + cx' + km = 0 \) is

- **Overdamped** when \( c^2 > 4km \)
- **Critically damped** when \( c^2 = 4km \)
- **Underdamped** when \( c^2 < 4km \)

Thus, this equation is overdamped.
11. Solve the system

\[ \begin{align*}
x' &= x + 2y \quad x(0) = 2 \\
y' &= 2x + y \quad y(0) = 0
\end{align*} \]

What is \( x(1) \), choose the closest answer.

(a) 14  
(b) 16  
(c) 18  
(d) 20 \( \rightarrow \) CORRECT 20.4534  
(e) 22  
(f) 24  
(g) 26  
(h) 28  
(i) 30  
(j) 32  
(k) 34

**Solution:**

\[
\begin{align*}
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{align*}
\]
12. Which of the following are eigenvalues of the matrix

\[ A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \]

(a) \( A \) has two or less eigenvalues.
(b) \( A \) has exactly three eigenvalues and they are: \(-1, 0, 1\)
(c) \( A \) has exactly three eigenvalues and they are: \(-1, 0, 2\)
(d) \( A \) has exactly three eigenvalues and they are: \(-1, 0, 3\) \( \rightarrow \text{CORRECT} \)
(e) \( A \) has exactly three eigenvalues and they are: \(-1, 1, 2\)
(f) \( A \) has exactly three eigenvalues and they are: \(-1, 1, 3\)
(g) \( A \) has exactly three eigenvalues and they are: \(0, 1, 2\)
(h) \( A \) has exactly three eigenvalues and they are: \(0, 1, 3\)
(i) \( A \) has exactly three eigenvalues and they are: \(0, 2, 3\)
(j) \( A \) has exactly three eigenvalues and they are: \(1, 2, 3\)
(k) \( A \) has four or more eigenvalues.

Solution: Eigenvalues are \(-1, 0, 3\).
13. Consider the initial value problem

\[ \begin{align*}
&x' = tx + y, \quad x(0) = 0 \\
y' = x + ty, \quad y(0) = 1
\end{align*} \]

Use Euler’s method with \( h = 0.5 \) to estimate \( x(1) \).

(a) 1.0
(b) 1.1 \quad \text{CORRECT} \quad 1.125
(c) 1.2
(d) 1.3
(e) 1.4
(f) 1.5
(g) 1.6
(h) 1.7
(i) 1.8
(j) 1.9
(k) 2.0

\textbf{Solution:}

<table>
<thead>
<tr>
<th>( n )</th>
<th>( t_n )</th>
<th>( X_n )</th>
<th>( K = F(t, X_n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>0 [ 1 ]</td>
<td>1 [ 0 ]</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.5 [ 1 ]</td>
<td>1.25 [ 1.0 ]</td>
</tr>
<tr>
<td>2</td>
<td>1.0</td>
<td>1.125 [ 1.5 ]</td>
<td></td>
</tr>
</tbody>
</table>
14. Let \( F(s) = \mathcal{L}\{f(t)\} \) where
\[
f(t) = \frac{e^{2t} - 1}{t}
\]
Find \( F(4) \), choose the closest answer.

(a) 0.0  
(b) 0.1  
(c) 0.2  
(d) 0.3  
(e) 0.4  
(f) 0.5  
(g) 0.6  
(h) 0.7 \( \rightarrow \text{CORRECT} \) \( \ln 2 \approx 0.69315 \)  
(i) 0.8  
(j) 0.9  
(k) 1.0

**Solution:**

\[
\begin{align*}
\mathcal{L}\left\{ \frac{e^{2t} - 1}{t} \right\} &= \int_s^\infty \mathcal{L}\{e^{2t} - 1\} \ d\sigma = \int_s^\infty \left( \frac{1}{\sigma - 2} - \frac{1}{\sigma} \right) \ d\sigma \\
&= \lim_{t \to \infty} \left[ \ln(\sigma - 2) - \ln \sigma \right]_t^s = \lim_{t \to \infty} \ln \left( \frac{\sigma - 2}{\sigma} \right) \big|_t^s \\
&= \lim_{t \to \infty} \left[ \ln \left( \frac{s}{s - 2} \right) - \ln \left( \frac{t}{t - 2} \right) \right] = \ln \left( \frac{s}{s - 2} \right)
\end{align*}
\]
15. Let $f(t) = \mathcal{L}^{-1}\{F(s)\}$ where

$$F(s) = \frac{1}{s^2 - 2s + 17}$$

Find $f(2)$, choose the closest answer.

(a) 0.0
(b) 1.6
(c) 1.8 \text{ \textbf{CORRECT}}
(d) 2.0
(e) 2.2
(f) 2.4
(g) 2.6
(h) 2.8
(i) 3.0
(j) 3.2

Solution: $f(t) = \frac{1}{4}e^t \sin 4t$
16. Which of the following equations is obtained by taking the Laplace transform of the initial value problem

\[ y'' + 2y' - ty = f(t), \quad y(0) = 0, y'(0) = 0 \]

where \( f(t) = \begin{cases} 1 & \text{if } 1 \leq t < 4 \\ 0 & \text{otherwise} \end{cases} \)

(a) \( (s^2 + s)Y(s) = \frac{e^{-s} - e^{-4s}}{s} \)

(b) \( (s^2 + s)Y(s) = \frac{e^{-4s} - e^{-s}}{s} \)

(c) \( (s^2 + s)Y(s) = e^{-s} - e^{-4s} \)

(d) \( (s^2 + s)Y(s) = e^{-4s} - e^{-s} \)

(e) \( Y'(s) - (s^2 + 2s)Y(s) = \frac{e^{-s} - e^{-4s}}{s} \)

(f) \( Y'(s) + (s^2 + 2s)Y(s) = \frac{e^{-s} - e^{-4s}}{s} \) \( \rightarrow \) CORRECT

(g) \( Y'(s) + (s^2 + 2s)Y(s) = e^{-s} - e^{-4s} \)

(h) \( Y'(s) + (s^2 + 2s)Y(s) = e^{-4s} - e^{-s} \)

(i) \( Y'(s) + (s^2 + 2s)Y(s) = \frac{e^s - e^{4s}}{s} \)

(j) \( Y'(s) - (s^2 + 2s)Y(s) = \frac{e^{-4s} - e^{-s}}{s} \)

(k) \( Y'(s) - (s^2 + 2s)Y(s) = e^s - e^{4s} \)

Solution:
17. Identify the true statements for the differential equation

\[ x^2(x^2 - 4)y'' + (x + 2)(\sin x)y' + x^2(x + 2)^3 y = 0 \]

I. There is exactly 1 singular point
II. There are exactly 2 singular points
III. There are 3 or more singular points
IV. 0 is a regular singular point
V. 0 is an irregular singular point

(a) I only
(b) II only
(c) III only
(d) I and IV only
(e) I and V only
(f) II and IV only → CORRECT
(g) II and V only
(h) III and IV only
(i) III and V only
(j) None of the above

Solution: The singular points are 0, 2. \( x = 0 \) is a regular singular point since

\[
P(x) = \frac{(x + 2) \sin x}{x^2(x^2 - 4)} = \frac{\sin x}{x^2(x - 2)} \\
Q(x) = \frac{(x + 2)^3}{x^2 - 4} = \frac{(x + 2)^2}{x - 2} \\
p(x) = xP(x) = \frac{(x + 2) \sin x}{x(x^2 - 4)} \\
q(x) = x^2Q(x) = \frac{x^2(x + 2)^3}{x^2 - 4} \\
p_0 = \lim_{x \to 0} p(x) = -\frac{1}{2} \\
q_0 = \lim_{x \to 0} q(x) = 0
\]
18. The point \( x = 0 \) is an ordinary point to the differential equation below and therefore there is a series solution \( y = \sum_{n=0}^{\infty} c_n x^n \). Find the recurrence relation for the coefficients \( c_n \).

\[(x - 2)y'' + (x - 1)y' + y = 0\]

(a) \( c_{n+2} = \frac{(n + 1)c_n}{2(n + 2)} \)

(b) \( c_{n+2} = \frac{(n - 1)c_n}{2(n + 2)} \)

(c) \( c_{n+2} = \frac{nc_n}{2(n + 2)(n + 1)} \)

(d) \( c_{n+2} = \frac{(n - 1)c_{n+1} + c_n}{2(n + 1)(n + 2)} \)

(e) \( c_{n+2} = \frac{(n - 1)c_{n+1} + nc_n}{2(n + 1)(n + 2)} \)

(f) \( c_{n+2} = \frac{(n + 2)c_{n+1} + nc_n}{2(n + 1)} \)

(g) \( c_{n+2} = \frac{(n - 1)c_{n+1} + nc_n}{2(n + 2)} \)

(h) \( c_{n+2} = \frac{(n + 1)c_{n+1} + c_n}{2(n + 2)} \)

(i) \( c_{n+2} = \frac{(n - 1)c_{n+1} + c_n}{2(n + 2)} \) \hspace{1cm} \rightarrow \text{CORRECT} \)

(j) \( c_{n+2} = \frac{(n - 1)c_{n+1} + c_n}{(n - 2)} \)

(k) None of the above

Solution: Substitute \( y = \sum_{n=0}^{\infty} c_n x^n \):

\[0 = (x - 2) \sum_{n=2}^{\infty} n(n - 1)c_n x^{n-2} + (x - 1) \sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n\]

\[= \sum_{n=2}^{\infty} n(n - 1)c_n x^{n-1} - 2 \sum_{n=2}^{\infty} n(n - 1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n\]

\[= \sum_{n=1}^{\infty} (n + 1) nc_{n+1} x^n - 2 \sum_{n=0}^{\infty} (n + 2)(n + 1)c_{n+2} x^n + \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} (n + 1)c_{n+1} x^n + \sum_{n=0}^{\infty} c_n x^n\]

\[= -4c_2 - c_1 + c_0 + \sum_{n=1}^{\infty} [-2(n + 2)(n + 1)c_{n+2} + (n + 1)(n - 1)c_{n+1} + (n + 1)c_n] x^n\]

Therefore the recursion relation is

\[c_{n+2} = \frac{(n + 1)(n - 1)c_{n+1} + (n + 1)c_n}{2(n + 2)(n + 1)}\]

\[= \frac{(n - 1)c_{n+1} + c_n}{2(n + 2)}\]
19. The point $x = 1$ is an ordinary point to the differential equation below and therefore there is a series solution $y = \sum_{n=0}^{\infty} c_n (x - 1)^n$. Find $c_2$.

$$(x - 2)y'' + (x - 1)y' + 2y = 0, \quad y(1) = 1, y'(1) = 0$$

(a) $-5/2$
(b) $-2$
(c) $-3/2$
(d) $-1$
(e) $-1/2$
(f) $0$
(g) $1/2$
(h) $1 \quad \rightarrow \text{CORRECT}$
(i) $3/2$
(j) $2$
(k) $5/2$

**Solution:** Set $t = x - 1$. The equation becomes

$$(t - 1)y'' + ty' + 2y = 0$$

The recurrence relation is

$$(n + 1)nc_{n+1} - (n + 2)(n + 1)c_{n+2} + (n + 2)c_n = 0$$

so $c_2 = c_0 = 1$. In fact, the series is

$$y = 1 + (x - 1)^2 + \frac{1}{3}(x - 1)^3 + \frac{1}{2}(x - 1)^4 + \frac{23}{60}(x - 1)^5 + \frac{46}{15}(x - 1)^6 + \cdots$$
20. $x = 0$ is a regular singular point for the differential equation below. Let $r_1 \geq r_2$ be the exponents for the series solution centered at $x = 0$ to the differential equation below.

$$x^2 y'' + \frac{2x^2}{\sin x} y' + (5x^2 - 6)y = 0$$

What is $\frac{r_1}{r_2}$?

(a) $-1/2$
(b) $-3/4$
(c) $-1/5$
(d) $-2/3 \rightarrow$ CORRECT
(e) 0
(f) 1
(g) 3/2
(h) 4/3
(i) 5
(j) 2
(k) None of the above

**Solution:**

$$p(x) = \frac{2x}{\sin x}$$
$$p_0 = \lim_{x \to 0} p(x) = 2$$
$$q(x) = 5x^2 - 6$$
$$q_0 = -6$$
$$r(r - 1) + 2r - 6 = 0$$
$$r = -3, 2$$
21. Identify the true statement for the differential equation

\[ x^2 y'' + \frac{x(x+1)}{2} y' - x^3 y = 0 \]

I. There is a series solution in the form \( y = \sum_{n=0}^{\infty} c_n x^n \)

II. There is a series solution in the form \( y = x^{1/2} \sum_{n=0}^{\infty} c_n x^n \)

III. There is a series solution in the form \( y = \sum_{n=0}^{\infty} c_n (x - 1)^n \)

IV. There is a series solution in the form \( y = \sum_{n=0}^{\infty} c_n (x - 2)^n \)

(a) Exactly one of these is true
(b) Exactly two of these is true
(c) All are true except I
(d) All are true except II
(e) All are true except III
(f) All are true except IV
(g) All are true \( \rightarrow \) CORRECT
(h) None of the above

Solution: The only singular point is at 0, which is regular. \( x = 1 \) and \( x = 2 \) are ordinary points and therefore there is a series solution centered at \( x = 1 \) and \( x = 2 \). The exponents at \( x = 0 \) are \( r_1 = \frac{1}{2} \) and \( r_2 = 0 \). Since \( r_1 - r_2 \notin \mathbb{Z} \), there is a Frobenius solution for each of these exponents.
22. One solution to the differential equation below is \( y_1 = x \).

\[
x^2 y'' + 2xy' - 2y = 0
\]

Let \( y(x) \) be the particular solution satisfying \( y(1) = 2, \ y'(1) = -4 \). What is \( y(\pi) \). Choose the closest answer.

(a) 0.0
(b) 0.1
(c) 0.2 \( \rightarrow \) CORRECT
(d) 0.3
(e) 0.4
(f) 0.5
(g) 0.6
(h) 0.7
(i) 0.8
(j) 0.9
(k) 1.0

**Solution:** Use reduction of order (or series) to find the general solution \( y = c_1 x + \frac{c_2}{x^2} \). The particular solution is \( y = \frac{2}{x^2} \). 
23. Let \( f(t) \) be the function with period \( 2\pi \) defined as

\[
f(t) = \begin{cases} 
2 & \text{if } 0 \leq x \leq \frac{\pi}{2} \\
0 & \text{if } \frac{\pi}{2} < x \leq 2\pi 
\end{cases}
\]

\( f(t) \) has a Fourier series and that series is equal to

\[
\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt
\]

Find \( \frac{a_3}{b_3} \). Choose the closest answer.

(a) \(-2.75\)
(b) \(-1.25\)
(c) \(-1.0 \rightarrow \text{CORRECT}\)
(d) \(-0.25\)
(e) \(1.25\)
(f) \(1.75\)
(g) \(2.5\)
(h) \(4.0\)
(i) \(6.25\)
(j) \(7.25\)

Solution:

\[
a_0 = 1 \\
a_n = \frac{2 \sin(n\pi/2)}{n\pi} \\
b_n = \frac{2 - 2 \cos(n\pi/2)}{n\pi} \\
a_3 = -\frac{2}{3\pi} \\
b_3 = \frac{2}{3\pi}
\]