Math 217 Spring 2001 Final Exam

9 May 2001

Notational Remark: In this exam, the symbols $\frac{\partial}{\partial x} y(x)$ **and D(y)(x) both mean "*the derivative of y(x) with respect to x, evaluated at x.*"**

No calculators of any kind. No "cheat sheets." Two Hours

1. Which one of the differential equations listed in answers (a) - (h) is *not* exact? (If each of the equations in answers (a)-(h) is exact, then choose answer (j). If two or more of the equations in answers (a)-(h) are not exact, then choose answer (i).)

a) $2\,x\,y^3\,dx + 3\,x^2\,y^2\,dy = 0$

b) $\cos(x+y)\,dx + \cos(x+y)\,dy = 0$

c) $e^{(x+2\,y)}\,dx + 2\,e^{(x+2\,y)}\,dy = 0$

d) $3\,dx + y\,dy = 0$

e) $(2\,x + 3\,y)\,dx + (3\,x + 2)\,dy = 0$

f) $(4\,x - y)\,dx + (6\,y - x)\,dy = 0$

g) $(2\,x\,y^2 + 3\,x^2)\,dx + (2\,x^2\,y + 4\,y^3)\,dy = 0$

h) $2\,y^2\,dx + (4\,x\,y + J_0(y))\,dy = 0$ (J_0 **is Bessel's function of order 0)**

i) At least two of the differential equations above are not exact.

j) All of the above differential equations are exact.

2. The solution to the initial value problem $\left(\frac{\partial}{\partial x}\mathrm{y}(x)\right)+\mathrm{y}(x)=\mathrm{y}(x)^2$, $\mathrm{y}(0)=\frac{1}{2}$ has what value at $x=\ln(2)$?

a) 0 b) $\frac{1}{3}$ c) $\frac{1}{2}$ d) $\frac{2}{3}$ e) $\frac{1}{6}$ f) $\frac{5}{6}$ g) $\frac{4}{3}$ h) 1 i) $\frac{7}{6}$ j) $\frac{11}{6}$

3. If $\mathrm{y}(x)$ is the solution to the initial value problem $\left(\frac{\partial^2}{\partial x^2}\mathrm{y}(x)\right)-5\left(\frac{\partial}{\partial x}\mathrm{y}(x)\right)+6\,\mathrm{y}(x)=0$, $\mathrm{y}(0)=2$, $\mathrm{D}(y)(0)=5$, then what is $\mathrm{y}(\ln(2))$?

a) 6 b) 7 c) 8 d) 9 e) 10 f) 11 g) 12 h) 13 i) 14 j) 15

4. The solution to the initial value problem

$$\left(\frac{\partial}{\partial x}\mathrm{y}(x)\right)+\frac{\mathrm{y}(x)}{1+x}=(1+x)^2,\ \mathrm{y}(0)=\frac{1}{4}$$

has what value at $x=1$?

a) $\frac{2}{5}$ b) $\frac{3}{5}$ c) $\frac{4}{5}$ d) 1 e) $\frac{5}{3}$ f) $\frac{8}{5}$ g) $\frac{9}{5}$ h) 2 i) $\frac{12}{5}$ j) $\frac{16}{5}$

5. Which one of the following expressions for $y(x)$ is a solution to the differential equation $4\left(\frac{\partial^2}{\partial x^2} y(x)\right) + 100\, y(x) = 0$?

a) $2\cos(5x) - 3\sin(5x)$

b) $3\sin(2x) + x\sin(2x)$

c) $4\,e^{(5x)}(\cos(x) + \sin(x))$

d) $e^{(10x)}(\cos(x) + 2\sin(x))$

e) $e^{(5x)} + e^{(10x)}$

f) $\cos(4x) - 3\sin(4x)$

g) $\cos(4x) - 2\sin(4x)$

h) $\cos(2x) - 2\sin(2x)$

i) $3\cos(10x) + \sin(10x)$

j) $2\cos(100x) + \sin(100x)$

6. If $y(x)$ is the solution to the initial value problem

$\left(\frac{\partial^2}{\partial x^2} y(x)\right) - 4\left(\frac{\partial}{\partial x} y(x)\right) + 4\, y(x) = 0$, $y(0) = 2$, $D(y)(0) = 5$, then what is $y(1)$?

a) e b) $2\,e$ c) $\frac{3}{e}$ d) e^2 e) $2\,e^2$

f) $3\,e^2$ g) $2 + e$ h) $1 + 2\,e$ i) $e^{(-2)}$ j) $\frac{2}{e^2}$

7. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$ denote the power series of the unique solution of the initial value problem $y'(x) = x^2 y(x) + 1,\ y(0) = 2$. What is the value of c_4?

a) 1 b) 1/2 c) 1/3 d) 2/3 e) 1/4 f) 3/4 g) 1/8 h) 3/8 i) 1/12 j) 5/12

8. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$ denote a power series of a solution of the differential equation $y''(x) + x\,y'(x) + y(x) = 0$. Then the coefficients satisfy which of the following recurrences for $1 \le n$?

a) $c_{n+1} = \dfrac{c_n}{n+1}$

b) $c_{n+1} = \dfrac{2 + c_n}{n+1}$

c) $c_{n+1} = -\dfrac{c_n}{n+1}$

d) $c_{n+1} = -\dfrac{2 + c_n}{n+1}$

e) $c_{n+1} = \dfrac{c_n}{n(n+1)}$

f) $c_{n+2} = \dfrac{c_n + c_{n-1}}{(n+1)(n+2)}$

g) $c_{n+2} = \dfrac{c_n - c_{n-1}}{n+1}$

h) $c_{n+2} = \dfrac{-c_n}{n+2}$

i) $c_{n+2} = \dfrac{c_n + c_{n-1}}{n+2}$

j) $c_{n+2} = \dfrac{c_n - c_{n-1}}{(n+1)(n+2)}$

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9. **When we seek a Frobenius series centered at 0 for a solution, which of the following equations is the indicial equation of the differential equation**

$$x\,y''(x) + y'(x) - \frac{2\,e^{x}\,y(x)}{x} = 0?$$

a) $r^2 - \frac{r}{2} = 0$ b) $r^2 - \frac{r}{2} - 1 = 0$ c) $r^2 + \frac{r}{2} = 0$ d) $r^2 + \frac{r}{2} - 1 = 0$ e) $\frac{r}{2} - 1 = 0$

f) $r^2 - 2 = 0$ g) $r^2 - \frac{r}{2} + 1 = 0$ h) $r^2 - r - 2 = 0$ i) $r^2 - r + 2 = 0$

j) There is no indicial equation because 0 is not a regular singular point of the equation.

10. Which one of the following Frobenius series might be a solution of the differential equation $x^2\left(\frac{\partial^2}{\partial x^2}y(x)\right)+2x\left(\frac{\partial}{\partial x}y(x)\right)+\frac{\cos(x)\,y(x)}{4}=0$ **with** $c_0 \neq 0$?

a) $y(x)=\sum_{n=0}^{\infty} c_n x^{\left(n-\frac{1}{2}\right)}$

b) $y(x)=\sum_{n=0}^{\infty} c_n x^{\left(n-\frac{1}{3}\right)}$

c) $y(x)=\sum_{n=0}^{\infty} c_n x^{\left(n-\frac{2}{3}\right)}$

d) $y(x)=\sum_{n=0}^{\infty} c_n x^{\left(n-\frac{3}{2}\right)}$

e) $y(x)=\sum_{n=0}^{\infty} c_n x^{\left(n-\frac{1}{4}\right)}$

f) $y(x)=\sum_{n=0}^{\infty} c_n x^{\left(n-\frac{1}{8}\right)}$

g) $y(x)=\sum_{n=0}^{\infty} c_n x^{\left(n-\frac{1}{16}\right)}$

h) $y(x)=\sum_{n=0}^{\infty} c_n x^{\left(n-\frac{1}{32}\right)}$

i) $y(x)=\sum_{n=0}^{\infty} c_n x^{\left(n-\frac{1}{64}\right)}$

j) $y(x)=\sum_{n=0}^{\infty} c_n x^{\left(n-\frac{64}{3}\right)}$

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11. Which expression is the inverse Laplace transform of $\dfrac{15\,s}{s^2 + 24\,s + 169}$?

a) $e^{(-13\,t)}\,(4\cos(5\,t) + 3\sin(5\,t))$

b) $3\,e^{(-13\,t)}\cos(12\,t) + 4\sin(12\,t)$

c) $e^{(-5\,t)}\,(3\cos(13\,t) - 4\sin(13\,t))$

d) $e^{(-5\,t)}\,(15\cos(12\,t) - 3\sin(12\,t))$

e) $e^{(-12\,t)}\,(15\cos(5\,t) - 3\sin(5\,t))$

f) $e^{(-24\,t)}\,(15\cos(5\,t) + 4\sin(5\,t))$

g) $3\,e^{(-13\,t)}\cos(5\,t)$

h) $\dfrac{3\,e^{(-12\,t)}\cos(5\,t)}{8}$

i) $\dfrac{3\,e^{(-5\,t)}\cos(12\,t)}{8}$

j) $e^{(-12\,t)}\,(15\cos(5\,t) - 36\sin(5\,t))$

12. Which expression is the inverse Laplace transform of $\dfrac{e^{(-5\,s)}}{s-2}$? (In the answers, the function u is the Heaviside function: u(t) = 1 for t > 0 and 0 for t < 0.)

a) $u(t-2)\,e^{(t-5)}$

b) $u(t-2)\,e^{(t+5)}$

c) $u(t+2)\,e^{(t-5)}$

d) $u(t+2)\,e^{(t+5)}$

e) $u(t-5)\,e^{(2\,t-5)}$

f) $u(t-5)\,e^{(2\,t-10)}$

g) $u(t+5)\,e^{(2\,t-5)}$

h) $u(t+5)\,e^{(t-10)}$

i) $u(t-10)\,e^{(t-5)}$

j) $u(t-10)\,e^{(t-10)}$

13. If $x(t)$ is the solution of

$$\left(\frac{\partial^2}{\partial t^2}x(t)\right) - 2\left(\frac{\partial}{\partial t}x(t)\right) + x(t) = e^t,\ x(0) = 0,\ D(x)(0) = 0$$

then what is $x(1)$?

a) $\frac{e}{2}$ b) e c) $1+e$ d) $\frac{1+e}{2}$ e) 2

f) $2+e$ g) $1+\frac{e}{2}$ h) $2e$ i) $1+2e$ j) $\frac{1}{2}+e$

14. The matrix A of the system

$$\frac{\partial}{\partial t}x(t) = x - 3y$$

$$\frac{\partial}{\partial t}y(t) = 2x - 4y$$

has an eigenvector v = [1,b] where b is an integer. If $A v = \lambda v$ then what is λ?

a) −4 b) -3 c) −2 d) -1 e) 0 f) 1

g) 2 h) 3 i) 4 j) insufficient information

15. If a solution to the system of the preceding problem satisfies $x(0) = 1$ and $y(0) = 2$ then what is $x(1)\,e^2$?

a) $1+2e$ b) $2-e$ c) $2+3e$ d) $3-2e$ e) $4+e$

f) $4-3e$ g) $5+e$ h) $4-5e$ i) $6+5e$ j) $5-6e$

16. If $x(t)$ is the solution of $\frac{\partial}{\partial t}x(t) = x(t) + \delta(t-1),\ x(0) = 0$ then what is $x(2)$?

a) 0 b) 1 c) e d) e^2 e) $e-1$

f) e^2-1 g) e^2-2 h) $e+2$ i) $\ln(2)$ j) $1+\ln(2)$

17. The matrix $\begin{bmatrix} 1 & 1 \\ -4 & -3 \end{bmatrix}$ has a chain of generalized eigenvectors {[1,-2], [2,b]}. What is b?

a) -4 b) -3 c) -2 d) -1 e) 0
f) 1 g) 2 h) 3 i) 4 j) 5

18. Suppose $f(t) = t$ for t in the interval $[0, \pi]$ and 0 in the interval $(\pi, 2\pi)$. Extend f so that it has period 2π. What is the coefficient of $\sin(2t)$ in the Fourier series of f?

a) 0 b) -1/2 c) 1 d) 3/2 e) -2 f) -5/2 g) 3 h)7/2 i) 4 j) 1/4

19. Let f be the periodization (of period 2) of the function that is -1 on the interval [-1,0) and 1 on the interval (0,1]. Which of the following statements <u>*best*</u> describes the Fourier series

$$\frac{a_0}{2} + \left(\sum_{k=1}^{\infty} (a_k \cos(k\pi t) + b_k \sin(k\pi t)) \right)$$

of f? (Read all proposed answers before choosing since more than one answer may be true. If that is the case, then choose the answer that conveys the greatest amount of true information about the Fourier coefficients.)

a) $a_k = 0$ for all $0 \le k$

b) $a_k = 0$ for all $0 < k$

c) $b_k = 0$ for all $0 \le k$

d) $b_k = 0$ for all positive even k

e) $b_k = 0$ for all positive odd k

f) $a_k = 0$ for all positive even k and $b_k = 0$ for all positive odd k

g) $b_k = 0$ for all $0 \le k$ and $a_k = 0$ for all positive even k

h) $b_k = 0$ for all $0 \le k$ and $a_k = 0$ for all positive odd k

i) $a_k = 0$ for all $0 \le k$ and $b_k = 0$ for all positive even k

j) $a_k = 0$ for all $0 \le k$ and $b_k = 0$ for all positive odd k

20. Let $f(t) = \sum_{n=1}^{\infty} \frac{\sin(n\,t)}{n^2}$. Which of the following expressions is the Fourier series expansion of a particular solution of the differential equation

$$\left(\frac{\partial^2}{\partial t^2} x(t)\right) + 5\,x(t) = f(t)?$$

a) $\frac{1}{10} + \left(\sum_{n=1}^{\infty} \frac{\cos(n\,t) + \sin(n\,t)}{5\,n^2}\right)$

b) $\frac{1}{10} + \left(\sum_{n=1}^{\infty} \frac{\cos(n\,t)}{5\,n^2}\right)$

c) $\sum_{n=1}^{\infty} \frac{\sin(n\,t)}{5\,n^2}$

d) $\frac{1}{2\,\pi} + \left(\sum_{n=1}^{\infty} \frac{\cos(n\,t) + \sin(n\,t)}{(5 - n^2)\,n^2}\right)$

e) $\frac{1}{2\,\pi} + \left(\sum_{n=1}^{\infty} \frac{\cos(n\,t)}{(5 - n^2)\,n^2}\right)$

f) $\sum_{n=1}^{\infty} \frac{\sin(n\sqrt{\pi}\,t)}{n^2}$

g) $\frac{1}{2\,\pi} + \left(\sum_{n=1}^{\infty} \frac{\cos(n\,t)}{5 + n^2}\right)$

h) $\sum_{n=1}^{\infty} \frac{\sin(n\,t)}{5 - n^2}$

i) $\sum_{n=1}^{\infty} \frac{\sin(n\,t)}{(5 - n^2)\,n^2}$

j) $\sum_{n=1}^{\infty} \frac{\sin(n\,\pi\,t)}{n^2}$