Name:
ID:
This exam has 20 multiple choice questions, worth 5 points each. You are allowed to use a scientific calculator and a 3 × 5 inch note card.

1. Which of the following pairs of vectors are parallel?

   I. \( \mathbf{A} = \langle 3, 2 \rangle \) and \( \mathbf{B} = \langle -9, -6 \rangle \)
   II. \( \overrightarrow{PQ} \) and \( \overrightarrow{OR} \), where \( P = (1, 1, 0), \ Q = (3, 5, 2), \ R = (-1, -2, -1), \) and \( O \) is the origin.
   III. \( \mathbf{A} = \langle 2, 1 \rangle \) and \( \overrightarrow{PQ} \), where \( P = (0, 1) \) and \( Q = (2, 1) \).

   (a) none of them
   (b) only I
   (c) only II
   (d) only III
   (e) only I and II
   (f) only I and III
   (g) only II and III
   (h) all of them

   (e) The vectors in I are parallel, since \(-3\mathbf{A} = \mathbf{B}\). For II, the vectors are \( \overrightarrow{PQ} = \langle 2, 4, 2 \rangle \) and \( \overrightarrow{OR} = \langle -1, -2, -1 \rangle \), which satisfies the equation \( \overrightarrow{PQ} = -2\overrightarrow{OR} \), so they are parallel. Finally, in III, \( \overrightarrow{PQ} = \langle 2, 0 \rangle \), which is not parallel to \( \mathbf{A} \).

2. Let \( \mathbf{A} \) and \( \mathbf{B} \) be vectors, where \( \|\mathbf{A}\| = 2, \|\mathbf{B}\| = 3 \), and where the angle between \( \mathbf{A} \) and \( \mathbf{B} \) is \( \pi/6 \) radians. What is \( \mathbf{A} \cdot \mathbf{B} \)?

   (a) \(-6\)
   (b) \(-3\sqrt{3}\)
   (c) \(-3\sqrt{2}\)
   (d) \(-3\)
   (e) 0
   (f) 3
   (g) \(3\sqrt{2}\)
   (h) \(3\sqrt{3}\)
   (i) 6

   (h) We know that \( \mathbf{A} \cdot \mathbf{B} = \|\mathbf{A}\|\|\mathbf{B}\| \cos \theta \), where \( \theta \) is the angle between them. So in this case, \( \mathbf{A} \cdot \mathbf{B} = (2)(3) \cos(\pi/6) = 3\sqrt{3} \).
3. What is the shortest distance between the point \( P = (1, 5) \) and the line passing through the points \((0,0)\) and \((1,1)\)?

(a) 0
(b) 1
(c) \(\sqrt{2}\)
(d) 2
(e) \(\sqrt{5}\)
(f) \(2\sqrt{2}\)
(g) 3
(h) 4
(i) 5
(j) \(\sqrt{26}\)

(f) One way to solve the problem is to use the vector \(A = \langle 0, 4 \rangle\) (the vector from \((1,1)\) to \((1,5)\)) and the vector \(B = \langle 1, 1 \rangle\) (a vector parallel to the line), and use the formula \(d = \frac{\|A \times B\|}{\|B\|} = \frac{4}{\sqrt{2}} = 2\sqrt{2}\).

4. In the following picture, \(A = \langle 2, 4, 5 \rangle\) and \(B = \langle 1, 1, 0 \rangle\). What is the vector \(C\)?

(a) \(\langle 3\sqrt{2}, 3\sqrt{2}, 0 \rangle\)
(b) \(\langle -1, 1, 5 \rangle\)
(c) \(\langle 6, 6, 0 \rangle\)
(d) \(\langle 3, 5, 5 \rangle\)
(e) \(\langle 1, -1, 0 \rangle\)
(f) \(\langle 3, 3, 0 \rangle\)
(g) \(\langle 5, -5, 3 \rangle\)
(h) \(\langle 1, 3, 5 \rangle\)
(i) \(\langle 2, -4, 5 \rangle\)
(j) \(\langle -5, 5, -2 \rangle\)

(b) In this picture, \(C\) is the part of \(A\) orthogonal to \(B\) (what we called \(\text{orth}_B A\) in class). It is given by the formula \(\text{orth}_B A = A - \text{proj}_B A = A - \left( \frac{A \cdot B}{\|B\|} \right) B = A - \frac{6}{\sqrt{2}} B = A - 3B = \langle -1, 1, 5 \rangle\).

5. Which one of the following parametric equations does NOT describe the line passing through the points \((2,0,1)\) and \((3,3,3)\)?

(a) \(X(t) = (2 + t, 3t, 1 + 2t)\)
(b) \( X(t) = (3 - t, 3 - 3t, 3 - 2t) \)
(c) \( X(t) = (2 + 2t, 6t, 1 + 4t) \)
(d) \( X(t) = (3 + t, 3 + 3t, 3 + 2t) \)
(e) \( X(t) = (1 + t, -3 + 3t, -1 + 2t) \)
(f) \( X(t) = (2 + 3t, 3t, 1 + 3t) \)

(f) The vector from \((2, 0, 1)\) to \((3, 3, 3)\) is \(\langle 1, 3, 2 \rangle\). Any line passing through these two points must be parallel to this vector. The equation for (f) can be written as \(X(t) = (2, 0, 1) + t\langle 3, 3, 3 \rangle\), from which we can see that it is parallel to the vector \(\langle 3, 3, 3 \rangle\). Since \(\langle 3, 3, 3 \rangle\) is not parallel to \(\langle 1, 3, 2 \rangle\), this equation can’t describe the line passing through \((2, 0, 1)\) and \((3, 3, 3)\).

6. Which one of the following equations describes the plane passing through the points \((0, 0, 1)\), \((1, 1, 1)\), and \((2, 0, 0)\)?
(a) \(x - y + 2z = 2\)
(b) \(x - y + z = 1\)
(c) \(x + 3y - z = 4\)
(d) \(x - y = 0\)
(e) \(z = 0\)
(f) \(x + y + z = 3\)
(g) \(x - 2y + z = 2\)
(h) The three points are collinear and therefore do not uniquely determine a plane.

(a) To get a normal vector, first find vectors between two pairs of points, for example \(\langle 1, 1, 0 \rangle\) goes from \((0, 0, 1)\) to \((1, 1, 1)\) and \(\langle 2, 0, -1 \rangle\) goes from \((0, 0, 1)\) to \((2, 0, 0)\). Then cross these vectors: \(\langle 1, 1, 0 \rangle \times \langle 2, 0, -1 \rangle = \langle -1, 1, -2 \rangle\). Then, using the general form \((X - P) \cdot N = 0\), where \(N = \langle -1, 1, -2 \rangle\) and \(P = (0, 0, 1)\), you get \(-x + y - 2(z - 1) = 0\), which can be rewritten as \(x - y + 2z = 2\).

7. The vectors \(A\), \(B\), and \(C\) lie flat on the page and point in the directions indicated below.

Each of the following answer choices lists three cross products involving \(A\), \(B\), and \(C\). In which answer do all three of the cross products point in the upward direction?

(a) \(A \times B\), \(B \times C\), and \(C \times A\)
(b) \(A \times B\), \(B \times C\), and \(A \times C\)
(c) \(A \times B\), \(C \times B\), and \(C \times A\)
(d) \(A \times B\), \(C \times B\), and \(A \times C\)
(e) \(B \times A\), \(B \times C\), and \(C \times A\)
(f) \( B \times A, B \times C, \) and \( A \times C \)

(g) \( B \times A, C \times B, \) and \( C \times A \)

(h) \( B \times A, C \times B, \) and \( A \times C \)

(f) This is an exercise in applying the right-hand rule. For example, \( B \times A \) points upward since you can point the fingers of your right hand toward \( B \) and curl them toward \( A \) while your thumb points up.

8. Let \( P = (2,1,0), Q = (0,0,1), R = (-2,1,1), \) and \( S = (6,3,-2) \). Three of these points are collinear (that is, there is a single line passing through all three points). Which three points are collinear?

(a) \( P, Q, \) and \( R \)
(b) \( P, Q, \) and \( S \)
(c) \( P, R, \) and \( S \)
(d) \( Q, R, \) and \( S \)

(b) \( \vec{PQ} = \langle -2, -1, 1 \rangle \) and \( \vec{PS} = \langle 4, 2, -2 \rangle \). These vectors are parallel \( (-2\vec{PQ} = \vec{PS}) \), which means that the points \( P, Q, \) and \( S \) are collinear.

9. Which one of the following vectors is parallel to the line of intersection of the planes \( 2x + y + z = 6 \) and \( x + z = 0 \)?

(a) \( \langle 2, 1, 1 \rangle \)
(b) \( \langle 1, 0, 1 \rangle \)
(c) \( \langle 3, 1, 2 \rangle \)
(d) \( \langle -3, 3, 3 \rangle \)
(e) \( \langle 6, 2, 1 \rangle \)
(f) \( \langle 1, -1, 1 \rangle \)
(g) \( \langle 0, 1, 0 \rangle \)
(h) \( \langle 2, -1, -1 \rangle \)

(d) A vector parallel to the line of intersection is perpendicular to both normal vectors. Thus, such a vector should be parallel to \( \langle 2, 1, 1 \rangle \times \langle 1, 0, 1 \rangle = \langle 1, -1, -1 \rangle \). The only vector among the choices that is parallel to \( \langle 1, -1, -1 \rangle \) is \( (d) \).

10. What is the shortest distance from the origin to the plane \( x + 2y + 2z = 5 \)?

(a) 0
(b) \( \frac{1}{5} \)
(c) \( \frac{1}{3} \)
(d) 1
(e) \( \frac{5}{3} \)
(f) \( \sqrt{3} \)
(g) 2
(h) 5/2
(i) 5

(e) The shortest distance from a point $Q$ to a plane is given by the formula $\frac{|\overrightarrow{PQ} \cdot N|}{\|N\|}$, where $P$ is any point on the plane. In this case, $(1,1,1)$ is a point on the plane, and a normal vector is $N = (1,2,2)$. So the distance is $\frac{|(-1,-1,-1)\cdot(1,2,2)|}{\|1,2,2\|} = \frac{5}{\sqrt{9}} = \frac{5}{3}$.

11. What is the area of the triangle with vertices $(0,0,1)$, $(1,1,1)$, and $(2,3,2)$?

(a) $\frac{1}{2}$
(b) $\sqrt{2}/2$
(c) $\sqrt{3}/2$
(d) 1
(e) $\sqrt{2}$
(f) 3/2
(g) $\sqrt{3}$
(h) 2
(i) 5/2
(j) 3

(c) If the points are called $P$, $Q$, and $R$, then the area of this triangle is half the area of the parallelogram spanned by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Thus the area of the triangle is $\frac{1}{2}\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \frac{1}{2}\|(1,1,0) \times (2,3,1)\| = \frac{1}{2}\|(1,-1,1)\| = \frac{\sqrt{3}}{2}$.

12. The motion of a particle is described by the parametric equation $X(t) = (t,t^2)$. How fast is the particle moving when $t = 1$?

(a) 0
(b) 1
(c) $\sqrt{2}$
(d) $\sqrt{3}$
(e) 2
(f) $\sqrt{5}$
(g) $\sqrt{6}$
(h) $\sqrt{7}$
(i) 3

(f) The velocity is $X'(t) = (1,2t)$, so the velocity vector at $t = 1$ is $X'(1) = (1,2)$. The speed is the norm of the velocity vector, which at $t = 1$ is $\sqrt{5}$.

13. Which one of the following equations describes the tangent line to the curve $X(t) = (e^t, 3t^2 + 2t)$ at the point $(1,0)$?
(a) \( L(t) = (1 + t, 2t) \)
(b) \( L(t) = (1 + t, 8t) \)
(c) \( L(t) = (1, 2t) \)
(d) \( L(t) = (1, 8t) \)
(e) \( L(t) = (1 + e^t, 6t + 2) \)
(f) \( L(t) = (1 + e, 2) \)
(g) \( L(t) = (1 + et, 2t) \)
(h) \( L(t) = (e^t, 6t + 2) \)

**a** The curve passes through the point \((1, 0)\) at \(t = 0\). The velocity is \(X'(t) = \langle e^t, 6t + 2 \rangle\), so \(X'(0) = \langle 1, 2 \rangle\). Thus a parametric equation for the tangent line is \(L(t) = (1, 0) + t \langle 1, 2 \rangle = (1 + t, 2t)\).

14. A particle travels one time around the helix \(X(t) = (\cos t, \sin t, t)\) (that is, as \(t\) goes from 0 to 2\(\pi\)). What is the distance that the particle has traveled?

(a) 1  
(b) \(\sqrt{2}\)  
(c) 2  
(d) \(2\sqrt{2}\)  
(e) \(2\pi\)  
(f) \(\sqrt{4\pi^2 + 1}\)  
(g) \(2\pi + 1\)  
(h) \((2\sqrt{2})\pi\)  
(i) \(4\pi\)  

**h** The velocity is \(X'(t) = \langle -\sin t, \cos t, 1 \rangle\), and the speed is \(\|X'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}\). The arclength is \(\int_0^{2\pi} \|X'(t)\| dt = \int_0^{2\pi} \sqrt{2} dt = 2\pi \sqrt{2}\).

15. The curve \(X(t) = (t^3 - t, t^2 - 1)\) passes through the origin twice, when \(t = -1\) and \(t = 1\). What is the angle at which the curve crosses itself at the origin?

(a) 0  
(b) \(\pi/6\)  
(c) \(\pi/4\)  
(d) \(\pi/3\)  
(e) \(\pi/2\)  
(f) \(2\pi/3\)  
(g) \(3\pi/4\)  
(h) \(5\pi/6\)  
(i) \(\pi\)
(e) The velocity is \(X'(t) = (3t^2 - 1, 2t)\). At \(t = -1\), the velocity vector is \(X'(-1) = (2, -2)\), and at \(t = 1\) it’s \(X'(1) = (2, 2)\). The angle of self-intersection equals the angle between the two velocity vectors, which is \(\pi/2\) since the two vectors are perpendicular.

16. Which one of the following Cartesian equations describes the parametric curve \(X(t) = (e^{2t}, e^{-t})\)?

(a) \(x^2 + y^2 = 1\)
(b) \(x^2 + 2y^2 = 1\)
(c) \(x + y = 2\)
(d) \(\ln x = y\)
(e) \(y = \sin x\)
(f) \(y = 2/x\)
(g) \(y = x^2\)
(h) \(xy^2 = 1\)

(h) If \(x = e^{2t}\) and \(y = e^{-t}\), then \(xy^2 = 1\).

17. Which one of the following functions is depicted in the contour plot below?

![Contour Plot](image)

(a) \(f(x, y) = x^2 + y^2\)
(b) \(f(x, y) = x + y^2\)
(c) \(f(x, y) = x - y^2\)
(d) \(f(x, y) = x^2 + y\)
(e) \(f(x, y) = x^2 - y\)
(f) \(f(x, y) = xy\)
(g) \(f(x, y) = x^2 - y^2\)
(h) \(f(x, y) = x/y\)
(i) \(f(x, y) = \sqrt{x^2 + y^2}\)

(b) The level curves of the function \(f(x, y) = x + y^2\) are given by equations \(x + y^2 = c\), or \(x = c - y^2\). These equations correspond to the curves in the contour plot. Note that the other possible choices don’t work; for example, the level curves of \(f(x, y) = x - y^2\) are parabolas pointing in the other direction than that on the contour plot.
18. In the following diagram, the vector \(d\) describes the displacement of a box. The vectors \(F\), \(G\), and \(H\) describe three different forces that act on the box as it moves in the direction of \(d\). Which of the following statements is true?

(a) The work done by \(F\) is greater than the work done by \(G\).
(b) The work done by \(H\) is greater than the work done by \(G\).
(c) All three forces do the same amount of work, and that amount is zero.
(d) All three forces do the same amount of work, and that amount is negative.
(e) All three forces do the same amount of work, and that amount is positive.

(e) Work is the dot product of the force vector and the displacement vector. For the vectors in this diagram, \(F \cdot d = G \cdot d = H \cdot d\). One way to see this is that, for example \(F \cdot d - G \cdot d = (F - G) \cdot d\), and \(F - G\) is orthogonal to \(d\).

19. If the motion of a particle is described by a parametric curve \(X(t) = (x(t), y(t), z(t))\), the speed of the particle is described by the function \(S(t) = \|X'(t)\|\). Which of the following is a correct formula for \(S'(t)\)?

(a) \(S'(t) = \|X''(t)\|\)
(b) \(S'(t) = X(t) \cdot X'(t)\)
(c) \(S'(t) = X'(t) \cdot X''(t)\)
(d) \(S'(t) = \frac{X'(t) \cdot X''(t)}{\|X'(t)\|}\)
(e) \(S'(t) = \frac{X(t) \cdot X'(t)}{\|X'(t)\|}\)
(f) \(S'(t) = \frac{\|X''(t)\|}{\|X'(t)\|}\)
(g) \(S'(t) = X'(t) \times X''(t)\)

(d) If \(S(t) = \|X'(t)\| = (X'(t) \cdot X'(t))^{1/2}\), then, using the ordinary chain rule for functions of one variable, you can compute

\[
S'(t) = \frac{1}{2}(X'(t) \cdot X'(t))^{-1/2} \left( \frac{d}{dt} [X'(t) \cdot X'(t)] \right)
\]
\[
= \frac{1}{2} \frac{2X'(t) \cdot X''(t)}{(X'(t) \cdot X'(t))^{1/2}}
\]
\[
= \frac{X'(t) \cdot X''(t)}{\|X'(t)\|}.
\]

In the second step, the product rule for the dot product is used.
20. The vectors $A$ and $B$ lie flat on the page and point in the directions indicated below.

Which of the following statements is true?

(a) $A \cdot B = 0$
(b) $A \cdot B > 0$
(c) $A \cdot B < 0$
(d) $A \times B = O$
(e) $A \cdot (A \times B) > 0$
(f) $A \cdot (A \times B) < 0$

(c) The angle between $A$ and $B$ is greater than $\pi/2$, so the dot product is negative. Note that the other statements are not true. The cross product is nonzero, since $A$ and $B$ are not parallel. And $A \cdot (A \times B) = 0$ since $A \times B$ is perpendicular to $A$. 