1. Find all the values of $\sin^{-1}(-i)$.

2. Find the determinant of the following matrix.

\[
\begin{pmatrix}
2 & 0 & 1 & 0 \\
0 & 3 & 0 & 1 \\
1 & 0 & 4 & 0 \\
0 & 1 & 0 & 5
\end{pmatrix}
\]

3. An $n \times n$ matrix $A$ is called skew-symmetric if $A^T = -A$. Show that $A^2$ is a symmetric matrix if $A$ is a skew-symmetric matrix.

4. Evaluate $\int_0^\infty e^{-2t} \sin 3tdt$.

5. Find the general solutions of the following differential equations $y'' + y = 2007$.

6. Consider a function

$f(x) = 2$, for $-\pi < x < 0$, $f(x) = 2007$, for $x = 0$, $f(x) = -2$, for $0 < x \leq \pi$

and its Fourier series $FS(f)$. Find the value of $FS(f)$ at $x = 0$.

7. Consider a function

$f(x) = 1$, for $-\pi < x \leq 0$, $f(x) = 0$, for $0 < x \leq \pi$

and its Fourier series $FS(f)$. Find the value of $\int_{-\pi}^{\pi} FS(f)(x)(\sin 2006x)dx$.

8. Find the direction of the line normal to the surface $x^{2007}y^{2007}z^{2007} + 1 = 0$ at the point $(1, 1, -1)$.

9. Let $S$ be the curved surface $x^2 + y^2 + z^2 = 4$ and $V = xyi + yzj + zk$, evaluate $\int \int_S (\nabla \times V) \cdot n \sigma$.

10. Evaluate.

$\int_0^\infty \frac{zd\lambda}{(1 + z^2)^3}$

11. Evaluate

$\lim_{n \to \infty} \frac{\Gamma(n + \frac{3}{2})}{\sqrt{n}\Gamma(n + 1)} = \frac{1}{\sqrt{\pi}}$

12. Let $a_{ij}$ be $1 \leq i, j \leq n$ skew symmetric tensor. Evaluate $\delta_{ij}a_{ij}$. **
** Choose 2 questions between questions 1 and question 6, and solve them. Each questions are 25 points.

1. If \(|z_1| < 1\) and \(|z_2| < 1\), show that

\[
\left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right| < 1.
\]

2. Let \(A\) be a \(2 \times 2\) real matrix defined as

\[
A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}.
\]

Consider \(A^{2007} = aA + bI_2\). Determine \(a\) and \(b\).

3. Write and solve the Euler equation to make the following integral stationary.

\[
\int_{x_1}^{x_2} \frac{yy'}{1 + yy'} dx
\]

4. Find the solution of the following differential equation by using the Laplace transform.

\[
y'' + 16y = 8 \cos 4t \quad y_0 = 0, y'_0 = 8
\]

5. Find the solution of the following differential equation.

\[
(D^2 - 6D + 9) y = 12xe^{3x}
\]

6. Evaluate \(\int_S (\nabla \times V) \cdot n \, d\sigma\) where \(S\) is the curved surface of the hemisphere \(x^2 + y^2 + z^2 = 9, \ z \geq 0\) and \(V = (xy^{2007} - yz^{2007})i + (yz^{2007} - zx^{2007})j + (zx^{2007} - xy^{2007})k\).
** Choose 2 questions between questions 1 and question 6, and solve them. Each questions are 30 points.

1. In the following, as you can see something is wrong. Find what and why is wrong.

   Consider
   \[
   \frac{1}{\sqrt{-1}} = \frac{1}{1}.
   \]
   Take square root,
   \[
   \sqrt{\frac{1}{\sqrt{-1}}} = \sqrt{\frac{-1}{1}}.
   \]
   And
   \[
   \frac{1}{\sqrt[3]{\sqrt{-1}}} = \frac{1}{\sqrt[3]{1}}.
   \]
   With cross product,
   \[
   1 = (\sqrt{-1})^2 = -1.
   \]

2. Consider a linear transformation \( A : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) defined as

   \[
   A = \begin{pmatrix}
   \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
   0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
   \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}
   \end{pmatrix}
   \]

   Show this is an orthogonal transformation and find the axes of the rotation.

3. Consider a curve joining two points \( x_1 \) and \( x_2 \) on \( x \)-axis with properly given length \( l \),
   fine the Euler equation whose solution is the equation of curve so that the surface of
   revolution formed by rotating the curve about the \( x \)-axis has minimum area.

4. Evaluate \( \int \int_S \mathbf{V} \cdot n d\sigma \) where \( S \) is the part of the surface \( z = 1 - x^2 - y^2 \) that is above
   the \( (x, y) \) plane and \( \mathbf{V} = xi + yj + zk \).

5. Consider the Bessel function \( J_\nu (x) \), evaluate \( J_\frac{\pi}{2} \) (show your work in detail).

6. Let \( a_{ij} \) be \( 1 \leq i, j \leq 2007 \) tensor defined as \( (i, j) \) entry of the following matrix.

   \[
   \begin{pmatrix}
   1 & 2 & 3 & \ldots & 2007 \\
   0 & 1 & 2 & \ldots & 2006 \\
   0 & 0 & 1 & \ldots & 2005 \\
   \vdots & \vdots & \vdots & \ddots & \vdots \\
   0 & 0 & 0 & \ldots & 1
   \end{pmatrix}
   \]

   Evaluate \( a_{ik}a_{kl} \).