Math 309 Midterm March 26, 2008 Key

Section 1.16 # 10 (10 points)
Section 2.2 # 34 (10 points)
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Section 4.1 # 32 (10 points)
Section 4.3 # 10 (10 points) and # 16 (10 points)
Section 4.4 # 14 (10 points)

10 (pg 101): 7% of a city's population moves to the suburbs each year, and 3% of the suburb's population moves to the city. In 2000, there were 800,000 people in the city and 500,000 in the suburbs. Set up the difference equation and estimate the populations in 2002.

Solution: If 7% of the city's population moves to the suburbs, then 93% stay in the city. If 3% of the suburb's population moves to the city, then 97% stay in the suburbs.

From:

\[
\begin{bmatrix}
.93 & .03 \\
.07 & .97
\end{bmatrix}
\]

to:

\[
\begin{bmatrix}
.93 & .03 \\
.07 & .97
\end{bmatrix}
\]

So \(M = \begin{bmatrix} .93 & .03 \\ .07 & .97 \end{bmatrix}\) and \(x_0 = \begin{bmatrix} 800,000 \\ 500,000 \end{bmatrix}\).

To find the populations in 2001, we compute \(MX_0\)

\[
x_1 = MX_0 = \begin{bmatrix} .93 & .03 \\ .07 & .97 \end{bmatrix} \begin{bmatrix} 800,000 \\ 500,000 \end{bmatrix} = \begin{bmatrix} .93(800,000) + .03(500,000) \\ .07(800,000) + .97(500,000) \end{bmatrix} = \begin{bmatrix} 759,000 \\ 591,000 \end{bmatrix}
\]

To find the populations in 2002, we compute \(MX_1\)

\[
x_2 = MX_1 = \begin{bmatrix} .93 & .03 \\ .07 & .97 \end{bmatrix} \begin{bmatrix} 759,000 \\ 591,000 \end{bmatrix} = \begin{bmatrix} .93(759,000) + .03(591,000) \\ .07(759,000) + .97(591,000) \end{bmatrix} = \begin{bmatrix} 722,100 \\ 577,900 \end{bmatrix}
\]

In 2002, 722,100 people live in the city and 577,900 live in the suburbs.
Section 2.2 #24) Guess the inverse of the matrix

\[ A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix} \]

Prove that your guess is correct

Solution:

Let \( A \) be as defined above and \( B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1/2 & 1/2 & 0 & \cdots & 0 \\ 0 & -1/3 & 1/3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1/n & 1/n \end{bmatrix} \)

and for \( j = 1, \ldots, n \), \( a_j = j^{th} \) column of \( A \), \( b_j = j^{th} \) column of \( B \), and \( e_j = j^{th} \) column of the identity matrix.

Interesting patterns arise for \( j = 1, \ldots, n-1 \):

\[ a_j = j(e_1 + \cdots + e_n) \]
\[ b_j = \frac{1}{j} e_j - \frac{1}{j+1} e_{j+1} \]

and \( b_n = \frac{1}{n} e_n \)

We can show, using these patterns, that \( AB = I \).

If we show that \( Ab_j = e_j \) for each \( j \). So,

\[ Ab_j = A \left( \frac{1}{j} e_j - \frac{1}{j+1} e_{j+1} \right) = \frac{1}{j} a_j - \frac{1}{j+1} a_{j+1} \]

\[ = (e_1 + \cdots + e_n) - (e_{j+1} + \cdots + e_n) = e_j \]

and \( Ab_n = A \left( \frac{1}{n} e_n \right) = \frac{1}{n} \alpha_n = e_n \).

We also note that the sum of \( B \)'s columns \( b_1 + \cdots + b_n = \frac{1}{j} e_j \). From this, we know that

\[ B a_j = j(Be_1 + \cdots + Be_n) = j(b_1 + \cdots + b_n) = j \left( \frac{1}{j} e_j \right) = e_j \]

Therefore \( BA = I \). Since \( BA = I \), \( B = A^{-1} \).
Let $f(t) = ax^3 + bx^2 + cx + d$ with $x_1, x_2, x_3$ distinct. Explain why $f(t)$ is a cubic polynomial, show that the coefficient of $t^3$ is non-zero, and find 3 points on the graph of $f$.

\[
V(t) = \begin{bmatrix}
1 & t & t^2 & t^3 \\
x_1 & x_1^2 & x_1^3 & \\
x_2 & x_2^2 & x_2^3 & \\
x_3 & x_3^2 & x_3^3 & 
\end{bmatrix}
\]

Solution: When we expand across the 1st row and find that $f(t) = \text{det}V = c_0 + c_1t + c_2t^2 + c_3t^3$.

$c_3 = \begin{vmatrix}
1 & x_1 & x_1^2 & \\
x_2 & x_2^2 & \\
x_3 & x_3^2 & 
\end{vmatrix} = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$

But, since $x_1, x_2, x_3$ are all distinct, each of the three factors above are nonzero. Therefore, $c_3 \neq 0$. Since $c_3$ is nonzero, the highest power of $f(t)$ will be 3, and $f(t)$ is a cubic polynomial. If we substitute any of $x_1, x_2, x_3$ for $t$, the matrix will have two identical (or linearly dependent) rows, and thus its determinant is zero. Since $f(t) = \text{det}V$, $f(t)$ will be zero at these points. 3 points on the graph of $f$ are $(x_1, 0), (x_2, 0), \text{and} (x_3, 0)$.

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Let $J$ be an $n \times n$ matrix of ones and $A = \begin{bmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{bmatrix}$. Confirm that $\text{det}A = (a-b)^n [a + (n-1)b]$ by:

a) Subtracting $R_2$ from $R_1$, $R_3$ from $R_2$, etc. and explain why this doesn't change $\text{det}A$.

b) With the matrix of $A$, add $C_1$ to $C_2$, $C_2$ to $C_3$, etc and explain why $\text{det}A$ doesn't change.

c) Find the determinant of the matrix in $b$. 

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Theorem 3 in Section 3.2 tells us that row replacement operations preserve determinants. Since we only performed row replacement operations, this has the same determinant as $A$.

Solution to $b)$ The matrix will be:

\[
\begin{bmatrix}
  a-b & 0 & 0 & \cdots & 0 \\
  0 & a-b & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  b & 2b & 3b & \cdots & a+(n-1)b \\
\end{bmatrix}
\]

Column operations are the same as doing row operations on $A^T$. So, this preserves the determinant of $A^T$. Since $\det A = \det A^T$, the determinant of $A$ is also preserved.

Solution to $c)$ The determinant of the above matrix $B$ has the same determinant as $A$. $B$ is a triangular matrix, so its determinant is the product of the diagonal elements. So,

\[
\det B = (a-b)^{n-1}(a+(n-1)b) = \det A
\]

Section 4.1 #32) If $H$ and $K$ are subspaces of $V$, show that $H \cap K$ is a subspace of $V$. Show that $H \cup K$ is not generally a subspace of $V$.

Solution to $a)$ To be a subspace of $V$, we must satisfy 3 conditions:

1) Zero vector: $H$ and $K$ both contain the zero vector of $V$ because they are subspaces of $V$. As this zero vector is common to both $H$ and $K$, we can say it is also in $H \cap K$, and thus $H \cap K$ contains the zero vector of $V$.

2) Closure under addition: Let $\vec{u}$ and $\vec{v}$ be vectors in $H \cap K$. Since $H \cap K$ is the collection of all vectors common to $H$ and $K$, $\vec{u}$ and $\vec{v}$ are contained within $H$ and $K$. Since $H$ is a subspace, $\vec{u} + \vec{v}$ is in $H$. Since $K$ is a subspace, $\vec{u} + \vec{v}$ is in $K$. Thus, $\vec{u} + \vec{v}$ is in $H \cap K$.

3) Closure under scalar multiplication: Let $\alpha$ be any scalar. Since $H \cap K$ is the collection of all vectors common to $H$ and $K$, $\alpha \vec{u}$ is contained within $H$ and $K$. Therefore, $\alpha \vec{u}$ is in $H \cap K$. Therefore, $H \cap K$ is a subspace of $V$.
3. Closure Under Scalar Multiplication: Let \( \mathbf{v} \) be a vector in \( H \) \& \( K \). \( \lambda \mathbf{v} \) is part of \( H \) and it is also part of \( K \).
Since \( H \) is a subspace, \( \lambda \mathbf{v} \) is in \( H \). Since \( K \) is a subspace, \( \lambda \mathbf{v} \) is in \( K \). Thus, \( \lambda \mathbf{v} \) is in \( H \cap K \). So \( H \cap K \) is closed under scalar multiplication.

Solution to b) \( H \cup K \) or \( H \cup K \), is the set of all elements in \( H \) and the set of all elements in \( K \), not just their intersection. Let \( H \) be the x-axis and let \( K \) be the y-axis. We know that \( H \) and \( K \) are both subspaces of \( \mathbb{R}^2 \). However, \( H \cup K \) is not a subspace because it is not closed under addition. If this still does not make sense, try adding an element of \( H \) (say the point (1,0)) to an element in \( K \) (say (0,1)). Their sum, (1,1), is neither an element of \( H \) nor \( K \), so \( H \cup K \) is not closed under addition.

![Diagram](image)

\( (0,1) \rightarrow \text{not on x-axis (H)} \)

\( (1,0) \rightarrow \text{or on y-axis (K)} \)

4.3 #10) Find a basis for \( \text{Nul} \ A \)

\[ A = \begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & -2 \\ 0 & 2 & -8 & 1 & 9 \end{bmatrix} \]

Solution: To find \( \text{Nul} \ A \), find the general solution to \( A \mathbf{x} = \mathbf{0} \)

\[ \begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & -2 \\ 0 & 2 & -8 & 1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 & 7 \\ 0 & 1 & -4 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \]
From \( \text{ref}(A) \), we find that:

\[
\begin{align*}
    x_1 - 5x_3 + 7x_5 &= 0 \\
    x_2 - 4x_3 + 6x_5 &= 0 \\
    x_4 - 3x_5 &= 0
\end{align*}
\]

Now we will solve for the variables of the pivot columns:

\[
\begin{align*}
    x_1 &= 5x_3 - 7x_5 \\
    x_2 &= 4x_3 - 6x_5 \\
    x_4 &= 3x_5
\end{align*}
\]

and put it into a vector:

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4 \\
    x_5
\end{bmatrix} =
\begin{bmatrix}
    5x_3 - 7x_5 \\
    4x_3 - 6x_5 \\
    x_3 \\
    3x_5 \\
    x_5
\end{bmatrix} =
\begin{bmatrix}
    5 \\
    4 \\
    1 \\
    0 \\
    0
\end{bmatrix} +
\begin{bmatrix}
    -7 \\
    -6 \\
    0 \\
    3 \\
    1
\end{bmatrix}
\]

Note that these 2 vectors are linearly independent. Thus, a basis for \( \text{null}(A) \) is

\[
\begin{bmatrix}
    5 \\
    4 \\
    1 \\
    0 \\
    0
\end{bmatrix},
\begin{bmatrix}
    -7 \\
    -6 \\
    0 \\
    3 \\
    1
\end{bmatrix}
\]

*Special Note: A basis for \( \text{null}(A) \) is simply these two vectors. The null space itself will be the possible linear combinations of these vectors, otherwise known as the span of these vectors.

**Section 4.3 #16** Find a basis for the vectors

\[
\begin{align*}
    \vec{v}_1 &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},
    \vec{v}_2 &= \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix},
    \vec{v}_3 &= \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix},
    \vec{v}_4 &= \begin{bmatrix} 5 \\ -3 \\ 3 \end{bmatrix},
    \vec{v}_5 &= \begin{bmatrix} 0 \\ -3 \\ -1 \end{bmatrix}
\end{align*}
\]
This is the exact same problem as finding a basis for $\text{Col } A$. We will now reduce $A$, identify its pivot columns, and form a basis from the original pivot columns of $A$. ($A$ is the matrix formed from $\frac{1}{3}$ thru $\frac{7}{6}$.

\[
\begin{bmatrix}
1 & -2 & 6 & 5 & 0 \\
0 & 1 & -1 & -3 & 3 \\
0 & -1 & 2 & 3 & -1 \\
1 & 1 & -1 & -4 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 0 & -1 & -2 \\
0 & 1 & 0 & -3 & 5 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The first 3 columns are pivot columns.

Taking the 1st 3 columns from the original matrix $A$, we see that the basis for $\text{Col } A$ is:

\[
\left\{
\begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix},
\begin{bmatrix}
2 \\
-1 \\
-1
\end{bmatrix},
\begin{bmatrix}
6 \\
-1 \\
2
\end{bmatrix}
\right\}
\]

**Section 4.4 # 44**

$B = \left\{ 1-t^2, t-t^2, 2-2t+t^2 \right\}$ is a basis for $M_2$. Find the coordinate vector of $p(t) = 3 + t - 6t^2$ relative to $B$.

**Solution**

Find $a$, $b$, and $c$ such that

\[
a(1-t^2) + b(t-t^2) + c(2-2t+t^2) = p(t) = 3 + t - 6t^2
\]

This produces the system of equations

\[
\begin{align*}
a + 2c &= 3 \\
b - 2c &= 1 \\
-a - b + c &= -6
\end{align*}
\]

We now reduce to find $a$, $b$, and $c$.

\[
\begin{bmatrix}
1 & 0 & 2 & 3 \\
0 & 1 & -2 & 1 \\
-1 & -1 & 1 & -6
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 0 & 7 \\
0 & 1 & 0 & -3 \\
0 & 0 & 1 & -2
\end{bmatrix}
\]

$a = 7$, $b = -3$, $c = -2$.

So $[p]_B = \begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix}$