Math 318  Exam II  March 28, '07

Name:__________________________________________

ID Number:_____________________________________

Do your work on this exam booklet. There is one page provided for each problem. Extra paper is available for scratch work. On questions 2, 3, and 4 give enough detail so that it is possible to follow your work.

<table>
<thead>
<tr>
<th>1</th>
<th>2a</th>
<th>2b</th>
<th>2c</th>
<th>3a</th>
<th>3b</th>
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1. (20 points) True or False.

1a. It is impossible to find 4 linearly independent vectors in \( \mathbb{R}^3 \).

1b. If \( T \) is a linear map of \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \) which is not the zero map then the function \( f(\vec{x}) = \| T\vec{x} \| \) does not have a maximum on the set \( \{ \vec{x} \in \mathbb{R}^3 : \| \vec{x} \| < 1 \} \).

1c. In the vector space of polynomials of degree at most 4 the polynomials \( p(x) = x^2 \), \( q(x) = x \) and \( r(x) = 2x^2 - 3x \) are linearly independent.

1d. The set in of all vectors \( [\vec{x}, \vec{y}] \) in \( \mathbb{R}^2 \) which satisfy the conditions \( xy = 0 \) and \( -1 \leq x - y \leq 1 \) is compact.

1e. If \( f(\vec{x}) \) is a continuous function on the set \( S = \{ [\vec{x}, \vec{y}] : x^2 + y^2 \leq 1 \leq \} \) then \( f \) has a maximum value on the set \( S \) and that maximum value is taken at a point \( \vec{a} \) where \( Df(\vec{a}) = 0 \).

Question 1 answers:  

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2. (35 points) Do 2 of 3.

2a. Perform a series of elementary operations on the rows of the matrix

\[
\begin{pmatrix}
2 & 1 \\
2 & -1 \\
4 & 4 \\
\end{pmatrix}
\]

to obtain the reduced echelon form of the matrix. What is the rank of the matrix?

2b. For which numbers \( a, b, \) and \( c \) does the system of equations

\[
\begin{align*}
2x + y &= 3 \\
2x - y &= 1 \\
4x + cy &= a
\end{align*}
\]
2x + y = a \\
2x - y = b \\
4x + 4y = c

have a solution.

2c. Suppose \( \vec{v_1}, \vec{v_2}, \) and \( \vec{v_3} \) are a basis of \( \mathbb{R}^3 \). Define \( \vec{w_1}, \vec{w_2}, \) and \( \vec{w_3} \) by
\[
\vec{w_1} = \vec{v_2} + \vec{v_3} \\
\vec{w_2} = \vec{v_1} + \vec{v_3} \\
\vec{w_3} = \vec{v_1} + \vec{v_2}.
\]
Must it be true \( \vec{w_1}, \vec{w_2}, \) and \( \vec{w_3} \) are a basis of \( \mathbb{R}^3 \)? Give a complete justification for your answer.

3. (35 points) Do 2 of 3.

3a. Find the maximum and minimum value of the function \( f(x, y) = x + 3y \) on the closed box \( \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\} \).

3b. Find and classify the critical points of the function \( f(x, y) = x^3 + y^2 - x - 2xy \).

3c. Use the method of Lagrange multipliers to maximize the function \( f(x, y) = xy^2 \) subject to the constraint \( x^2 + y^2 = 3 \).

4. (10 points) Do 1 of 2.

4a. Suppose \( \vec{u} \) and \( \vec{v} \) are linearly independent vectors in \( \mathbb{R}^4 \). For which numbers \( \alpha, \beta \) are the two vectors \( \vec{w} = \vec{u} + \vec{v} \) and \( \vec{z} = \alpha \vec{u} + \beta \vec{v} \) linearly independent.

4b. Recall that if \( T \) is a linear map from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) then the norm of \( T \), \( \| T \| \), is defined to be
\[
\| T \| = \max_{\| x \|=1} \| T x \|.
\]
Show that if \( S \) is another linear map from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) then
\[
\| S - T \| \leq \| S \| + \| T \|.
\]
1. 2a. Perform a series of elementary operations on the rows of the matrix

\[
\begin{pmatrix}
2 & 1 \\
2 & -1 \\
4 & 4 \\
\end{pmatrix}
\]

to obtain the reduced echelon form of the matrix. What is the rank of the matrix?
2b. For which numbers $a$, $b$, and $c$ does the system of equations

\begin{align*}
2x + y &= a \\
2x - y &= b \\
4x + 4y &= c
\end{align*}

have a solution.
2c. Suppose $\vec{v}_1, \vec{v}_2,$ and $\vec{v}_3$ are a basis of $\mathbb{R}^3$. Define $\vec{w}_1, \vec{w}_2$, and $\vec{w}_3$.

\[
\vec{w}_1 = \vec{v}_2 + \vec{v}_3 \\
\vec{w}_2 = \vec{v}_1 + \vec{v}_3 \\
\vec{w}_3 = \vec{v}_1 + \vec{v}_2
\]

Must it be true $\vec{w}_1, \vec{w}_2$, and $\vec{w}_3$ are a basis of $\mathbb{R}^3$? Give a complete justification for your answer.
3a. Find the maximum and minimum value of the function $f(x, y) = x + 3y$ in the closed box \( \{ (x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1 \} \).
3b. Find and classify the critical points of the function \( f(x, y) = x^3 + y^2 - x - 2xy \).
3c. Use the method of Lagrange multipliers to maximize the function $f(x, y) = xy^2$ subject to the constraint $x^2 + y^2 = 3$. 
4a. Suppose \( \vec{u} \) and \( \vec{v} \) are linearly independent vectors in \( \mathbb{R}^4 \). For which numbers \( \alpha, \beta \) are the two vectors \( \vec{w} = \vec{u} + \vec{v} \) and \( \vec{z} = \alpha \vec{u} + \beta \vec{v} \) linearly independent.
4b. Recall that if $T$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ then the norm of $T$, $\|T\|$ is defined to be

$$
\|T\| = \max_{\|x\|=1} \|Tx\|.
$$

Show that if $S$ is another linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ then

$$
\|S - T\| \leq \|S\| + \|T\|.
$$