# Math 318 Final Exam Solutions May 8, '07

**I Do 4 of 5, 15 points each.**

**1.** Find the orthogonal (perpendicular) projection of the vector $\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ onto the subspace $\mathbf{V}$ of $\mathbb{R}^3$ described by the equation $x_1 + x_3 = 0$. Find a vector $\mathbf{y}$ in $\mathbf{V}$ so that the vector $\mathbf{x} - \mathbf{y}$ is perpendicular to $\mathbf{V}$.

**A** We need an orthogonal basis of $\mathbf{V}$. $\mathbf{V}$ is a 2 dimensional plane so any two linearly independent vectors in $\mathbf{V}$ will be a basis and if they are orthogonal we will have what we want. We could work analytically, but by inspection and experimentation we can find two such vectors; for instance $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. The formula from the book now tells us that

$$\begin{aligned} \mathrm{Proj}_{\mathbf{V}}\mathbf{x} &= \frac{\langle \mathbf{x}, \mathbf{e}_1 \rangle}{\|\mathbf{e}_1\|^2}\mathbf{e}_1 + \frac{\langle \mathbf{x}, \mathbf{e}_2 \rangle}{\|\mathbf{e}_2\|^2}\mathbf{e}_2. \\ &= \frac{1}{2}(-1)\mathbf{e}_1 + 0\mathbf{e} \\ &= \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \end{aligned}$$

The decomposition of any vector $\mathbf{w}$ into something in $\mathbf{V}$ plus something perpendicular to $\mathbf{V}$ is

$$\begin{aligned} \mathbf{w} &= \mathrm{Proj}_{\mathbf{V}}\mathbf{w} + (\mathbf{w} - \mathrm{Proj}_{\mathbf{V}}\mathbf{w}) \\ &= \text{in} \quad + \quad \text{perp} \end{aligned}$$

so we set $\mathbf{y} = \mathrm{Proj}_{\mathbf{V}}\mathbf{w}$ for the second part, that gives $\mathbf{x} - \mathbf{y} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$ which is perpendicular to $\mathbf{V}$.

**2.** Evaluate the integral $\int_R f\,dA$ for

$$f\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = x^2 y$$

$$R = \left\{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) : x \geq 1/2,\ y \geq 0,\ x^2 + y^2 \leq 1\right\}.$$

**A**

$$\begin{aligned}
\int_R f dA &= \int_{1/2}^{1}\left(\int_0^{\sqrt{1-x^2}} x^2 y\,dy\right)dx \\
&= \int_{1/2}^{1}\left(\frac{1}{2}x^2y^2\Big|_{y=0}^{y=\sqrt{1-x^2}}\right)dx \\
&= \int_{1/2}^{1}\frac{1}{2}x^2\left(\sqrt{1-x^2}\right)^2 dx \\
&= \frac{1}{2}\int_{1/2}^{1}(x^2 - x^4)dx \\
&= \frac{47}{960}
\end{aligned}$$

**3.** Find the volume of the region in $\mathbb{R}^3$ bounded above by the surface $z + x^2 + y^2 = 12$, bounded below by the surface $z = \sqrt{x^2 + y^2}$, and to the right of the plane $x = 0$ ( i.e., in the region where $x > 0$).

**A** We go to cylindrical coordinates. The top and bottom of the region are $z = 12 - r^2$ and $z = r$. The intersection of those surfaces is when $12 - r^2 = r$, and the positive solution of that equation is $r = 3$. The description of the region is cylindrical coordinates is $0 < r < 3, -\frac{\pi}{2} < \theta < \frac{\pi}{2}, r < z < 12 - r^2$. So

$$\begin{aligned}
\text{Volume} &= \int_{-\pi/2}^{\pi/2}\left(\int_0^1\left(\int_r^{12-r^2} r\,dz\right)dr\right)d\theta \\
&= \int_{-\pi/2}^{\pi/2}\left(\int_0^3((12 - r^2 - r)r)dr\right)d\theta \\
&= \int_{-\pi/2}^{\pi/2}\frac{99}{4}d\theta \\
&= \frac{99}{4}\pi
\end{aligned}$$

**4.** Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}$. Find $\det A$. Find $A^{-1}$.

**A**

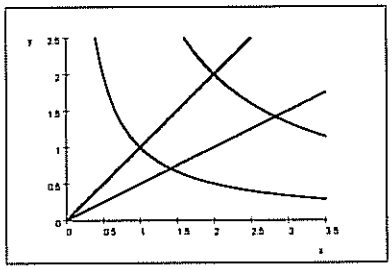
$$\det A = -12$$

$$A^{-1} = \frac{-1}{12}\begin{pmatrix} 0 & 0 & 6 \\ 0 & 6 & -3 \\ 4 & -4 & 0 \end{pmatrix}$$

**5.** Evaluate $\int_R f\,dA$ where

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{x^2+y^2}{xy}$$

and $R$ is bounded by the four curves $xy = 1$, $xy = 4$, $x = y$, $x = 2y$.



$R$

**A** We will change coordinates to $u = xy, v = x/y$. The integration in the new coordinates will be on the region $1 \le u \le 4, 1 \le v \le 2$. We need to write $x$ and $y$ in terms of $u$, $v$. We have $x = \sqrt{uv}, y = \sqrt{u/v}$. The change of variables is given by $g$.

$$g\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \sqrt{uv} \\ \sqrt{u/v} \end{pmatrix}$$

$$Dg = \begin{pmatrix} \frac{\partial}{\partial u}\sqrt{uv} & \frac{\partial}{\partial v}\sqrt{uv} \\ \frac{\partial}{\partial u}\sqrt{u/v} & \frac{\partial}{\partial v}\sqrt{u/v} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \\ \frac{1}{2\sqrt{uv}} & -\frac{1}{2}\sqrt{\frac{u}{v^3}} \end{pmatrix}$$

$$\left|\det Dg\begin{pmatrix} u \\ v \end{pmatrix}\right| = \left|\det\begin{pmatrix} \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \\ \frac{1}{2\sqrt{uv}} & -\frac{1}{2}\sqrt{\frac{u}{v^3}} \end{pmatrix}\right| = \left|-\frac{1}{2v}\right| = \frac{1}{2v}$$

$$fg\begin{pmatrix} u \\ v \end{pmatrix} = \frac{\sqrt{uv}^2 + \sqrt{u/v}^2}{\sqrt{uv}\sqrt{u/v}} = v + \frac{1}{v}$$

$$\int_R f dA = \int\int_{\substack{1\le u\le 4 \\ 1\le v\le 2}} \left(v + \frac{1}{v}\right)\frac{1}{2v}dA_{(u,v)} = \int_1^4\left(\int_1^2 \frac{1}{2} + \frac{1}{2v^2}dv\right)du = \int_1^4 \frac{3}{4}du = \frac{9}{4}$$

**II Do 2 of 3, 10 points each.** $x^2 + y^2 + \left(\frac{1}{4}(36 - 2x - 3y)\right)^2$

**1.** Find the point on the plane $2x + 3y + 4z = 36$ closest to the origin.

**A** We need to minimize $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + y^2 + z^2$ subject to $2x + 3y + 4z = 36$. (Minimizing the square of the distance to the origin will still give us the closest point.) We could use the method of Lagrange multipliers and it would work, relatively quickly. Instead we will use the second equation to eliminate $z$. Thus we want to minimize

$$g(x,y) = x^2 + y^2 + \left(\frac{1}{4}(36 - 2x - 3y)\right)^2 = x^2 + y^2 + \left(9 - \frac{1}{2}x - \frac{3}{4}y\right)^2$$

$$\frac{\partial}{\partial x}g = 2x + 2\left(9 - \frac{1}{2}x - \frac{3}{4}y\right)\left(-\frac{1}{2}\right) = \frac{5}{2}x - 9 + \frac{3}{4}y$$

$$\frac{\partial}{\partial y}g = 2y + 2\left(9 - \frac{1}{2}x - \frac{3}{4}y\right)\left(-\frac{3}{4}\right) = \frac{25}{8}y + \frac{3}{4}x - \frac{27}{2}$$

Setting the partial derivatives equal zero, clearing fractions, and rearranging, we need to solve

$$10x + 3y = 36$$
$$6x + 25y = 108$$

So $x = 72/29$ and $y = 108/29$, and hence, for the point on the plane, $z = -144/29$. You can use the second derivative test to see this gives is a minimum, or note, geometrically, that the function $g$ clearly has one min and no max.

**2.** Use the method of Lagrange multipliers to find the <u>maximum</u>, if there is one, <u>and minimum</u>, if there is one, of the function $f\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = x + 2y$ subject to the condition that $x^2 + 4y^2 = 16$.

**A** Set $g = x^2 + 4y^2 - 16$. We want to find points where $Df$ and $Dg$ are parallel; i.e., there is a $\lambda$ so that $(1,2) = \lambda(2x, 8y)$. This gives $\lambda = \frac{1}{2x}$ and $\lambda = \frac{2}{8y}$ and hence $x = 2y$. Hence

$$(2y)^2 + 4y^2 = 16$$
$$y^2 = 2$$
$$y = \pm\sqrt{2} \text{ and hence}$$
$$x = \pm\sqrt{8}$$

The choice $+, +$ gives $\sqrt{2} + \sqrt{8}$, the max; the choice $-, -$ gives $-\sqrt{2} - \sqrt{8}$, the min.

**3.** What condition, if any, must the numbers $a$, $b$, and $c$ satisfy in order for the system of equations

$$\begin{aligned} 2x + 3y + z &= a \\ x + 2y + z &= b \\ 3x + 4y + z &= c \end{aligned}$$

to have a solution? If that condition is satisfied will the solution be unique?

**A** We set up the augmented matrix and proceed by row reduction to obtain

$$\left(\begin{array}{ccc||c} 2 & 3 & 1 & a \\ 1 & 2 & 1 & b \\ 3 & 4 & 1 & c \end{array}\right) \to \left(\begin{array}{ccc||c} 0 & -1 & -1 & a-b \\ 1 & 2 & 1 & b \\ 0 & -2 & -2 & c-3b \end{array}\right) \to \left(\begin{array}{ccc||c} 0 & -1 & -1 & a-2b \\ 1 & 2 & 1 & b \\ 0 & 0 & 0 & c-3b-2(a-2b) \end{array}\right)$$

hence to have a solution we must have $0 = c - 3b - 2(a - 2b) = c + b - 2a$. The solution is not unique.

Details weren't asked for; however, we showed that the third equation is a linear combination of the first two. Hence if you take any solution and add to it numbers $x, y$, and $z$ which make the left hand sides of the first to equations 0 it will leave the two left hand sides unchanged and, because the left hand side of the third is a linear combination (twice the first, minus the second) of the other two, it will also be unchanged. Thus adding the triple $x, y, z$ to your solution will give a new solution. Given any $x$ it is always possible to find $y$ and $z$ so that this works out.

**III Do 5 of 5, 4 points each.**

**1.** The matrix $A$ and its inverse $A^{-1}$ are:

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & -2 & 6 \\ -3 & 4 & 5 \end{pmatrix}, \quad A^{-1} = \frac{1}{88}\begin{pmatrix} 34 & -6 & -20 \\ 23 & -17 & 2 \\ 2 & 10 & 4 \end{pmatrix}.$$

Use $A^{-1}$ to solve the system of equations

$$\begin{aligned} x + 2y + 4z &= 6 \\ x - 2y + 6z &= 6 \\ -3x + 4y + 5z &= 6 \end{aligned}$$

**A** The solution is

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= A^{-1}\begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix} \\ &= \frac{1}{88}\begin{pmatrix} 34 & -6 & -20 \\ 23 & -17 & 2 \\ 2 & 10 & 4 \end{pmatrix}\begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix} \\ &= \begin{pmatrix} \frac{6}{11} \\ \frac{6}{11} \\ \frac{12}{11} \end{pmatrix} \end{aligned}$$

**2.** Suppose $A$ is a linear mapping from $\mathbb{R}^4$ to $\mathbb{R}^5$. The range of $A$, Range($A$), is defined to be all vectors $\mathbf{y}$ in $\mathbb{R}^5$ which can be written as $\mathbf{y} = A\mathbf{x}$ for some $\mathbf{x}$ in $\mathbb{R}^4$; that is, Range($A$) $= \{A\mathbf{x} : \mathbf{x} \text{ in } \mathbb{R}^4\}$. Is Range($A$) a subspace of $\mathbb{R}^5$? Justify your answer.

**A** Yes, we need to check three things:

- $A\mathbf{0} = \mathbf{0}$ so $\mathbf{0}$ is in Range($A$),
- If $\mathbf{y}$ is in Range($A$) then $\mathbf{y} = A\mathbf{x}$ and $\lambda\mathbf{y} = \lambda A\mathbf{x} = A\lambda\mathbf{x}$ so $\lambda\mathbf{y}$ is in Range($A$),
- If $\mathbf{y}$, $\mathbf{y}'$ are in Range($A$) then $\mathbf{y} = A\mathbf{x}$, $\mathbf{y}' = A\mathbf{x}'$, hence
$$\mathbf{y} + \mathbf{y}' = A\mathbf{x} + A\mathbf{x}' = A(\mathbf{x} + \mathbf{x}')$$
so $\mathbf{y} + \mathbf{y}'$ is in Range($A$).

**3.** Suppose $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are non-zero vectors in $\mathbb{R}^4$ which are pairwise orthogonal (i.e. $\mathbf{x} \perp \mathbf{y}$, $\mathbf{x} \perp \mathbf{z}$, and $\mathbf{z} \perp \mathbf{y}$). Must it be true that the three vectors are linearly independent? Justify your answer.

**A** Yes. Suppose $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} = \mathbf{0}$. Taking the inner product of both sides of the equation with $\mathbf{x}$ gives

$$\begin{aligned} \langle a\mathbf{x} + b\mathbf{y} + c\mathbf{z}, \mathbf{x} \rangle &= \langle \mathbf{0}, \mathbf{x} \rangle \\ a\langle \mathbf{x}, \mathbf{x} \rangle + b\langle \mathbf{y}, \mathbf{x} \rangle + c\langle \mathbf{z}, \mathbf{x} \rangle &= \langle \mathbf{0}, \mathbf{x} \rangle \\ a\|\mathbf{x}\|^2 + 0 + 0 &= 0 \end{aligned}$$

so $a\|\mathbf{x}\|^2 = 0$. $\mathbf{x}$ is not the zero vector so $\|\mathbf{x}\|^2$ is not 0, so $a$ is 0. Similarly $b = c = 0$.

**4.** Give a vector of length 1 pointing in the direction in which the function $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xy + x^2z - 2yz$ is increasing most rapidly at the point $\mathbf{p} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. Show your work.

**A** The gradient points in the direction of most rapid increase. So the answer is

$$\frac{1}{\|\nabla f(\mathbf{p})\|}\nabla f(\mathbf{p}) = \frac{1}{\sqrt{11}}\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

**5** Immediately after the definition of derivative the book states, "This says that $D\mathbf{f}(\mathbf{a})$ is the best linear approximation to the function $\mathbf{f} - \mathbf{f}(\mathbf{a})$ at $\mathbf{a}$.". For the given function $\mathbf{f}$ and base point $\mathbf{a}$ give the explicit formula for that "best approximation". Explain briefly and clearly what is "best" about it.

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + 2y \\ 2xy \\ x - 1 \end{pmatrix}; \quad \mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

**A**

$$f(\mathbf{a}) = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

$$df = \begin{pmatrix} 2x & 2 \\ 2y & 2x \\ 1 & 0 \end{pmatrix}$$

$$df(\mathbf{a}) = \begin{pmatrix} 2 & 2 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}$$

The approximation is

$$f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) \approx df(\mathbf{a})\mathbf{h}$$

$$f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right) - \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \approx \begin{pmatrix} 2 & 2 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \text{ or}$$

$$(*) = f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right) - \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \approx 0 \text{ if } \mathbf{h} \text{ is small.}$$

This is the "best" approximation because if we replace $df(\mathbf{a})$ in $(*)$ with any other $3 \times 2$ matrix then that new expression will be small when $\mathbf{h}$ is small but it will not be nearly as small as $(*)$ is (for the same $\mathbf{h}$).