Math 5021, Fall 2013

Assignment 4, 1 October 2013

PROGRESS: This week we are finishing Chapter 2 and will probably start Chapter 3.
DUE: This assignment is due Tuesday, 8 October at the start of class.

PROBLEMS:

1. We proved the Maximum Modulus Theorem in class: If f is a nonconstant function in $\mathrm{Hol}(\Omega)$ for some region Ω then $|f(z)|$ cannot have a local maximum in Ω. A strict analog for the minimum modulus fails. If $f(z_0) = 0$ then $|f(z)|$ has a local minimum at z_0. Formulate and prove a theorem which says that this is the only obstacle to a Minimum Modulus Theorem.

2. Use your Minimum Modulus Theorem to give a proof of the Fundamental Theorem of Algebra.

3. Suppose $a \in \mathbb{D}$. Define $B_a(z)$ by

$$B_a(z) = \frac{z-a}{1-\bar{a}z}.$$

   (a) Note that B_a is holomorphic in $\{z : |z| \leq 1\}$. Show that $|B_a(z)| = 1$ if $|z| = 1$. Conclude that $|B_a(z)| \leq 1$ in $\mathbb{D}$.

   (b) Use this function to prove a Schwartz type lemma for functions f which are holomorphic in $\mathbb{D}$ and are zero at $z = a$. (i.e. $B_a(z)$ will play the role in the proof that was played by z.)

4. Continue the notation of the previous problem. Suppose f is holomorphic in $\bar{\mathbb{D}}$ and that $|f(z)| = 1$ when $|z| = 1$. Show that there is a finite set $\{a_i\}_{i=1}^N \subset \mathbb{D}$ (which may include repetitions) so that

$$f(z) = \prod_{i=1}^{N} B_{a_i}(z).$$

5. Using the previous result if you like, show that an entire function which has modulus one on the unit circle must be a monomial.