

CHORDAL AND SEQUENTIALLY COHEN-MACAULAY CLUTTERS

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ABSTRACT. We extend the definition of chordal from graphs to clutters. The resulting family generalizes both chordal graphs and matroids, and obeys many of the same algebraic and geometric properties. Specifically, the independence complex of a chordal clutter is shellable, hence sequentially Cohen-Macaulay; and the circuit ideal of a certain complement to such a clutter has a linear resolution. Minimal non-chordal clutters are also closely related to obstructions to shellability, and we give some general families of such obstructions, together with a classification by computation of all obstructions to shellability on 6 vertices.

The shelling of the independence complex is constructed by extending the definitions of shedding face and k -decomposability to non-pure complexes. The resulting tool to construct shellings may be of independent interest.

1. INTRODUCTION

A *clutter* \mathcal{C} is a hypergraph such that no edge of \mathcal{C} is properly contained in any other edge. For example, any graph is a clutter, as is any d -uniform hypergraph. There is a dual relationship between simplicial complexes and clutters, as follows: Given any clutter \mathcal{C} , there is an *independence complex* $I(\mathcal{C})$ which has faces consisting of all subsets of $V(\mathcal{C})$ containing no edge of \mathcal{C} . Given any simplicial complex Δ , there is a *non-face clutter* $\mathcal{C}(\Delta)$ on the same vertex set with edges consisting of the minimal subsets of $V(\Delta)$ which are not faces. Clearly $I(\mathcal{C}(\Delta)) = \Delta$ and $\mathcal{C}(I(\mathcal{C})) = \mathcal{C}$.

The non-face clutter of Δ is perhaps most familiar via the *Stanley-Reisner ring* of Δ :

$$R[\Delta] \triangleq R[x_1, \dots, x_n] / (x_{i_1} x_{i_2} \cdots x_{i_k} : \{x_{i_1}, \dots, x_{i_k}\} \text{ an edge of } \mathcal{C}(\Delta)),$$

where $V(\Delta) = \{x_1, \dots, x_n\}$. The ideal in this construction is known as the *edge ideal* or *circuit ideal* of $\mathcal{C}(\Delta)$.

Recently, a number of papers [10, 12, 16, 19, 28, 30] have asked what one can say about the algebraic and topological combinatorics of Δ from the structure of $\mathcal{C} = \mathcal{C}(\Delta)$. A particularly successful case has

been that where $\mathcal{C} = G$ is a chordal graph. In this case, the independence complex $I(G)$ is vertex decomposable [30], hence shellable [28] and sequentially Cohen-Macaulay [16], while the edge ideal of the complement \bar{G} has a linear resolution [17]. Moreover, the chordal graphs are closely related to the family of graphs with every induced subcomplex of the (flag) independence complex shellable [30]; and the complements of chordal graphs are exactly the graphs with edge ideal having a linear resolution [17].

In the current paper, our purpose is to introduce a family of clutters, which we call *chordal clutters*, which satisfy similar properties. Chordal clutters generalize several previously studied families, including chordal graphs, circuit clutters of matroids, and acyclic clutters. We will prove:

Theorem 1.1. *If \mathcal{C} is a chordal clutter then the independence complex $I(\mathcal{C})$ is shellable and hence sequentially Cohen-Macaulay.*

The technique that we use may be of broader interest. Provan and Billera [25] introduced k -decomposability of pure simplicial complexes, and Björner and Wachs [5, Section 11] generalized vertex decomposability (i.e., 0-decomposability) to non-pure complexes. In order to prove Theorem 1.1, we need to generalize k -decomposability to non-pure complexes for all k . The concrete condition we give for a *shedding face* may be useful for showing shellability in broader contexts than chordal clutters.

We also prove:

Theorem 1.2. *Let \mathcal{C} be a d -uniform chordal clutter. Then the circuit ideal of the complement of \mathcal{C} has a linear resolution.*

As previously mentioned, there are interesting converse or partial converse results to both Theorem 1.1 and Theorem 1.2 when \mathcal{C} is a chordal graph. We discuss the possibility of finding such results for general chordal clutters. We relate Theorem 1.1 to obstructions to shellability via both examples and computational results, but conclude that an interesting description of clutters with circuit ideal having linear resolution (i.e., a converse to Theorem 1.2) is unlikely.

The structure of the papers is as follows. In Section 2 we present the necessary background material. In Section 3 we define k -decomposability for non-pure simplicial complexes, and extend many of the theorems proved in the pure case by Provan and Billera [25]. In Section 4 we define simplicial vertices in clutters, which leads naturally to a definition of a chordal clutter. We give several examples of such clutters, including chordal graphs and the circuit clutters of matroids. In Section 5 we prove that the independence complex of any chordal clutter

is shellable, as well as some other results about shellability of independence complexes of more general graphs and clutters. In Section 6 we recall the basic facts about linear resolutions, Alexander duality, and their relationship. We then use the combinatorial Alexander dual to prove that the circuit ideal of a certain uniform complement of a chordal clutter has linear resolution. We close in Section 7 by relating forbidden minors to chordality with obstructions to shellability. We give several infinite families of these forbidden minors, and characterize by computation with GAP [18] all obstructions to shellability on 6 or fewer vertices that contain no non-shellable link.

2. BACKGROUND

2.1. Simplicial complex and clutter notation. An *abstract simplicial complex* Δ on a vertex set V is a set of subsets of V (called *faces* of Δ) such that each subset of a face of Δ is itself a face of Δ . We do not require that elements of V are faces of Δ . A maximal face is called a *facet*, and a d -dimensional face (having cardinality $d + 1$) is called a d -face.

The *join* of two simplicial complexes Δ_1 and Δ_2 on disjoint vertex sets V_1 and V_2 is a complex $\Delta_1 * \Delta_2$ on vertex set $V_1 \cup V_2$ with faces $\{\sigma_1 \cup \sigma_2 : \sigma_i \text{ a face of } \Delta_i\}$.

A *clutter* \mathcal{C} on a vertex set V is a set of subsets of V (called *circuits* or *edges* of \mathcal{C}) such that if e_1 and e_2 are distinct circuits of \mathcal{C} then $e_1 \not\subseteq e_2$. Clutters have also been studied under the name *Sperner family*. To avoid confusion with 1-faces of a simplicial complex, we will usually prefer the term “circuit” over “edge”. A d -circuit is a circuit consisting of exactly d vertices, and a clutter is *d-uniform* if every circuit has exactly d vertices. The *maximal circuit cardinality* of \mathcal{C} is the largest cardinality of any circuit in \mathcal{C} , and similarly for the minimal circuit cardinality.

An *independent set* of \mathcal{C} is a set containing no circuit. Clutters and simplicial complexes are linked via the independence complex $I(\mathcal{C}) = \{\sigma \subseteq V : \sigma \text{ is an independent set of } \mathcal{C}\}$, and via the clutter of minimal non-faces $\mathcal{C}(\Delta)$. As previously mentioned we have $\mathcal{C}(I(\mathcal{C})) = \mathcal{C}$ and $I(\mathcal{C}(\Delta)) = \Delta$.

There are two degenerate simplicial complexes on V : the simplicial complex $\{\}$ with no faces, and the simplicial complex $\{\emptyset\}$ with only the empty set as a face. Following [12], we call $\{\}$ the (-1) -simplex and $\{\emptyset\}$ the (-1) -sphere. Notice that $\mathcal{C}(\{\}) = \emptyset$, while $\mathcal{C}(\{\emptyset\}) = \{\{v\} : v \in V\}$.

Nondegenerate simplicial complexes admit a *geometric realization*, a geometric simplicial complex with the same face incidences. When we use geometric or topological terms such as dimension, homotopy type, etc. to discuss simplicial complexes, we are referring to the geometric realization.

All clutters and simplicial complexes considered are finite.

2.2. Deletions and contractions, links. Given a simplicial complex Δ , two kinds of subcomplex are of particular interest. If σ is a face of Δ , then $\Delta \setminus \sigma$ is obtained from Δ by removing all faces that contain σ from the set system. The *star* of σ is $\text{star}_\Delta \sigma = \{\text{faces containing } \sigma\}$, and the *link* of σ is the simplicial complex on vertex set $V(\Delta) \setminus \sigma$ with faces

$$\text{link}_\Delta \sigma = \{\tau : \sigma \cap \tau = \emptyset, \sigma \cup \tau \text{ is a face of } \Delta\},$$

Thus $\text{link}_\Delta \sigma$ is $\text{star}_\Delta \sigma$ with all vertices of σ deleted, while $\text{star}_\Delta \sigma = (\text{link}_\Delta \sigma) * \sigma$.

Given a clutter \mathcal{C} , there are two ways of removing a vertex that are of interest. Let $v \in V(\mathcal{C})$. The *deletion* $\mathcal{C} \setminus v$ is the clutter on the vertex set $V(\mathcal{C}) \setminus \{v\}$ with circuits $\{e : e \text{ a circuit of } \mathcal{C} \text{ with } v \notin e\}$. The *contraction* \mathcal{C}/v is the clutter on the vertex set $V(\mathcal{C}) \setminus \{v\}$ with circuits the minimal sets of $\{e \setminus \{v\} : e \text{ a circuit of } \mathcal{C}\}$. Thus, $\mathcal{C} \setminus v$ deletes all circuits containing v , while \mathcal{C}/v removes v from each circuit containing it (and then removes any redundant circuits).

A clutter \mathcal{D} obtained from \mathcal{C} by repeated deletion and/or contraction is called a *minor* of \mathcal{C} . If \mathcal{D} is obtained only by deletions we call it an *induced subclutter* on vertex set $V(\mathcal{D})$. If \mathcal{D} is obtained only by contractions we call it a *contraction* of \mathcal{C} . Notice that if all the vertices contained in a circuit are contracted, then what remains is the clutter $\{\emptyset\}$. It is straightforward to prove that if $v \neq w$ are vertices then $(\mathcal{C} \setminus v) \setminus w = (\mathcal{C} \setminus w) \setminus v$, $(\mathcal{C}/v)/w = (\mathcal{C}/w)/v$, and $(\mathcal{C} \setminus v)/w = (\mathcal{C}/w) \setminus v$.

Example 2.1. A clutter \mathcal{C} is a *matroid circuit clutter* if it satisfies the *weak circuit exchange property*: that if e_1 and e_2 are circuits of \mathcal{C} containing a common vertex v , then there is some circuit e_3 such that $e_3 \subseteq (e_1 \cup e_2) \setminus \{v\}$. In this case the deletion and contraction operations $\mathcal{C} \setminus v$ and \mathcal{C}/v are the usual deletion and contraction in a matroid without loops or coloops. See [24] for additional background and other definitions of matroids.

We collect the relationships between simplicial complex operations and clutter operations in the following lemma.

Lemma 2.2. *Let \mathcal{C}, \mathcal{D} be clutters, and let v be a vertex of $V(\mathcal{C})$.*

- (1) $I(\mathcal{C}/v) = \text{link}_{I(\mathcal{C})} v$.
- (2) $I(\mathcal{C}\setminus v) = I(\mathcal{C})\setminus v$, considered as a simplicial complex on $V(\mathcal{C})\setminus v$.
- (3) If \mathcal{C}' consists of the minimal sets of $\mathcal{C} \cup \{\sigma\}$ (where σ is an independent set), then $I(\mathcal{C}') = I(\mathcal{C}) \setminus \sigma$.
- (4) $I(\mathcal{C} \dot{\cup} \mathcal{D}) = I(\mathcal{C}) * I(\mathcal{D})$.

There are obvious dual versions of these which relate a simplicial complex Δ with its minimal non-face clutter $\mathcal{C}(\Delta)$.

Example 2.3. If G is a graph (i.e., a clutter with every circuit having cardinality 2), then let $N[v] = \{v \text{ and all its neighbors}\}$. Then $G \setminus v$ is the induced subgraph on $V \setminus \{v\}$, while G/v is the induced subgraph $G \setminus N[v]$ together with a singleton circuit for each neighbor w to v . In particular, $I(G/v) = I(G \setminus N[v])$.

Remark 2.4. Contraction in the clutter sense in a graph G should not be confused with the distinct notion of contraction in the circuit matroid of G !

Remark 2.5. In a graph, every contraction operation can be expressed (up to singleton circuits) as repeated deletion operations, deleting each vertex in $N[v]$. This does not hold true in general clutters. For example, if \mathcal{C} consists of a single circuit of cardinality 3, then every deletion has no circuits while \mathcal{C}/v has a circuit.

2.3. Shellable and Cohen-Macaulay complexes. We recommend the unfamiliar reader to [4, 5] for background on shellability, and to [3, 27, 6] for Cohen-Macaulay and sequentially Cohen-Macaulay simplicial complexes, but give a brief review here.

Let k be any field or the ring of integers. A simplicial complex is *Cohen-Macaulay* if, for every face σ (including $\sigma = \emptyset$), we have $H_i(\text{link}_\Delta \sigma, k) = 0$ for $i < \dim(\text{link}_\Delta \sigma)$. An equivalent condition is for the Stanley-Reisner ring of Δ to be a Cohen-Macaulay ring, and there is also an equivalent topological condition which makes no reference to the face structure of Δ . Examples of complexes that are Cohen-Macaulay (over any k) include any triangulation of a sphere.

Every Cohen-Macaulay simplicial complex Δ is *pure*, i.e., every pair of facets have the same dimension; but many interesting clutters have non-pure independence complex. The *pure d -skeleton* of Δ is the sub-complex generated by all faces of dimension d . We say that a complex is *sequentially Cohen-Macaulay over k* if the pure d -skeleton is Cohen-Macaulay (over k) for all d . Once again, there are equivalent ring-theoretic and topological conditions for the sequentially Cohen-Macaulay property. Any pure sequentially Cohen-Macaulay complex is Cohen-Macaulay.

A simplicial complex Δ is *shellable* if its facets fit together nicely. The precise definition is that there is an ordering $\sigma_1, \dots, \sigma_m$ of the facets of Δ such that the intersection of σ_i with the complex generated by $\sigma_1, \dots, \sigma_{i-1}$ is $(\dim \sigma_i - 1)$ -dimensional. When possible, we avoid this definition and work through the condition of k -decomposability introduced in Section 3, or through the characterization of a shelling in Theorem 3.5.

Any shellable complex is sequentially Cohen-Macaulay over any k , and we view shellability as a combinatorial condition for a complex to be sequentially Cohen-Macaulay. Since the results we prove will be independent of the field or ring k , we henceforth suppress k from our notation.

A *linear resolution* of an ideal I in a ring R is a minimal free resolution of R/I satisfying certain properties – the exact definition will not be important to us, as we work through a characterization of Eagon and Reiner [11]. As the details are somewhat complicated, and only required in Section 6, we defer further discussion to this section.

2.4. Notation. We will use letters \mathcal{C} and \mathcal{D} for clutters, except in the special case where the clutter is a graph, when we defer to convention and use G . Other calligraphic letters such as \mathcal{F} will denote families of objects. Vertices in clutters will be denoted by v, w , etc; while circuits (edges) will be denoted with the letter e . We use upper case Greek letters such as Δ and Σ for simplicial complexes. Inside a simplicial complex, we use letters v and w for vertices, and lower case Greek letters such as σ and τ for faces.

3. k -DECOMPOSABLE COMPLEXES

Provan and Billera [25] introduced a definition of k -decomposability for pure complexes. For $k = 0$ these are known as vertex decomposable complexes, and the definition of vertex decomposable complexes was extended to non-pure complexes by Björner and Wachs. We now give an analogous extension of k -decomposability to non-pure complexes for $k > 0$.

Definition 3.1. Let Δ be a simplicial complex on vertex set V . Then a face σ is called a *shedding face* if every face τ of $\text{star}_\Delta \sigma$ satisfies the following exchange property: for every $v \in \sigma$ there is a $w \in V \setminus \tau$ such that $(\tau \cup \{w\}) \setminus \{v\}$ is a face of Δ .

Remark 3.2. An equivalent condition to the exchange property of Definition 3.1 is the following: no facet of $(\text{star}_\Delta \sigma) \setminus \sigma$ is a facet of $\Delta \setminus \sigma$.

Remark 3.3. In the case where σ is a single vertex, our definition of shedding vertex specializes to that of Björner and Wachs [5, Section 11]. In the case that Δ is pure, our definition specializes to that of Provan and Billera [25, Definition 2.1].

The main fact we will prove about shedding faces is the following generalization of [29, Lemma 6]:

Lemma 3.4. *If σ is a shedding face for Δ such that both $\Delta \setminus \sigma$ and $\text{link}_\Delta \sigma$ are shellable, then Δ is shellable.*

Before proving Lemma 3.4, we recall the characterization of shellings due to Björner and Wachs [4]. Given a shelling $\sigma_1, \sigma_2, \dots, \sigma_m$ of a simplicial complex Δ , let Δ_i be the subcomplex generated by facets $\sigma_1, \sigma_2, \dots, \sigma_i$. Then the *restriction map* of the shelling is

$$(3.1) \quad R(\sigma_i) = \{x : \sigma_i \setminus \{x\} \text{ a face of } \Delta_{i-1}\}.$$

Björner and Wachs prove:

Theorem 3.5. (Björner and Wachs, [4, Proposition 2.5]) *Let $\sigma_1, \dots, \sigma_m$ be an ordering of the facets of a simplicial complex Δ , and let R be a map $\{\sigma_1, \dots, \sigma_m\} \rightarrow \Delta$. The ordering is a shelling order with restriction map R if and only if $R(\sigma_i) \not\subseteq \sigma_j$ for $i > j$ and every face of Δ contains $R(\sigma_i)$ for exactly one i .*

Proof. (of Lemma 3.4) Since $\text{star}_\Delta \sigma = \text{link}_\Delta \sigma * \sigma$, we have that $\text{star}_\Delta \sigma$ is shellable if and only if $\text{link}_\Delta \sigma$ is. Then the shelling order is obtained by taking the shelling order τ_1, \dots, τ_m of $\Delta \setminus \sigma$, followed by the shelling order ρ_1, \dots, ρ_n of $\text{star}_\Delta \sigma$. The shedding face condition implies that no face of $\text{star}_\Delta \sigma$ is a facet of $\Delta \setminus \sigma$, hence the two lists of facets are disjoint.

We prove this is a shelling by an application of Theorem 3.5, as follows. Define R as in (3.1), and let $R_{\Delta \setminus \sigma}$ and $R_{\text{link}_\Delta \sigma}$ be the restriction maps associated with the shelling τ_1, \dots, τ_m of $\Delta \setminus \sigma$ and $\rho_1 \setminus \sigma, \dots, \rho_n \setminus \sigma$ of $\text{link}_\Delta \sigma$. It is obvious that $R(\tau_i) = R_{\Delta \setminus \sigma}(\tau_i)$, and it follows since $(\text{star}_\Delta \sigma) \setminus \sigma \subseteq \Delta \setminus \sigma$ that $R(\rho_j) = \sigma \cup R_{\text{link}_\Delta \sigma}(\rho_j \setminus \sigma)$. Since every $R(\rho_j)$ contains σ , no $R(\rho_i) \subseteq \tau_i$ for any i ; and every face of Δ is contained in exactly one $R(\tau_i)$ or $R(\rho_j)$, depending on whether or not σ is contained in the face. \square

We see that the definition of shedding face can be viewed as a tool to build up a shelling of Δ by “sorting” the facets of Δ .

Definition 3.6. A simplicial complex Δ is recursively defined to be k -decomposable if either Δ is a simplex or else has a shedding face σ with $\dim \sigma \leq k$ such that both $\Delta \setminus \sigma$ and $\text{link}_\Delta \sigma$ are k -decomposable. We

consider the degenerate complexes $\{\}$ and $\{\emptyset\}$ to be k -decomposable for $k \geq -1$.

Definition 3.6 obviously extends the definition of vertex decomposability and pure k -decomposability.

Many of the theorems proved by Provan and Billera go through straightforwardly for our definition. Most interesting from the author's perspective is:

Theorem 3.7. *A d -dimensional (not necessarily pure) simplicial complex Δ is shellable if and only if it is d -decomposable.*

Proof. Lemma 3.4 gives the “if” direction.

For the other direction, take $\sigma_1, \dots, \sigma_m$ to be a shelling order of Δ and R to be its restriction map as in (3.1). Then $R(\sigma_m)$ is a shedding face, since $\text{star}_\Delta R(\sigma_m) = \sigma_m$ [4, Lemma 2.4], and since $\sigma_m \setminus \{x\}$ is a face of Δ_{m-1} for all $x \in R(\sigma_m)$ by definition. The result follows by induction. \square

Theorem 3.7 tells us that k -decomposability gives a hierarchical structure on the family of shellable complexes. Every k -decomposable complex also $(k+1)$ -decomposable, and every shellable d -dimensional complex is j -decomposable for at least one $j \leq d$.

Some other theorems that extend easily are:

Proposition 3.8. *If Δ is k -decomposable, then $\text{link}_\Delta \tau$ is k -decomposable for every face τ of Δ .*

Proof. Entirely similar to [25, Proposition 2.3]. \square

The following is stronger than [25, Proposition 2.4]:

Proposition 3.9. *Δ_1 and Δ_2 are k -decomposable if and only if $\Delta_1 * \Delta_2$ is k -decomposable.*

Proof. The “only if” direction is entirely similar to [25, Proposition 2.4].

Conversely, suppose that σ is a shedding face of $\Delta_1 * \Delta_2$, with $\sigma_i = V(\Delta_i) \cap \sigma$. Let τ_1 be any facet of Δ_1 which contains σ_1 , and τ_2 similarly for Δ_2 and σ_2 . Then every face of the form $\rho_1 \dot{\cup} \tau_2$ with $\sigma_1 \subset \rho_1$ satisfies the shedding face exchange property, and (since τ_2 is a facet) every face ρ_1 of Δ_1 containing σ_1 satisfies the exchange property. Thus σ_1 and by symmetry σ_2 are shedding faces for Δ_1 and Δ_2 . Clearly σ_1 and σ_2 are each of dimension $\leq k$.

Next, we notice that $\text{link}_\Delta \sigma = \text{link}_{\Delta_1} \sigma_1 * \text{link}_{\Delta_2} \sigma_2$, and by induction each of $\text{link}_{\Delta_1} \sigma_1$ and $\text{link}_{\Delta_2} \sigma_2$ is k -decomposable. Finally, $\Delta_1 \setminus \sigma_1 = \text{link}_{\Delta \setminus \sigma} \tau_2$, which is k -decomposable by Proposition 3.8. \square

There are other interesting results in [25], most notably the bound on diameter of a pure k -decomposable complex, but we content ourselves with extending these few. Our main application of k -decomposability will come in Section 5, where we use it to prove that the independence complex of a chordal clutter is shellable.

Remark 3.10. Simon [26, Section 2.3] has introduced “clean ideal trees,” an extension of k -decomposability via commutative algebra, however our concrete condition for a shedding face seems to better lend itself to constructing shellings.

Question 3.11. *If \mathcal{C} is a clutter with maximal edge cardinality of d , then what is the minimal k for which $I(\mathcal{C})$ is k -decomposable?*

We prove the following lemma for use in Section 6. A more general refinement of [4, Theorem 2.9] to k -decomposability might be an interesting future direction.

Lemma 3.12. *Let Δ be a vertex decomposable simplicial complex. Then the s -skeleton of Δ is vertex decomposable for any s .*

Proof. Let $\Delta^{(s)}$ denote the s -skeleton of a simplicial complex. Clearly, $\text{link}_{\Delta^{(s)}} v = (\text{link}_{\Delta} v)^{(s-1)}$, while $\Delta^{(s)} \setminus v = (\Delta \setminus v)^{(s)}$. Thus, if Σ_n denotes the n -simplex, then $\Sigma_n^{(s)} \setminus v = \Sigma_{n-1}^{(s)}$, while $\text{link}_{\Sigma_n^{(s)}} v = \Sigma_{n-1}^{(s-1)}$. Hence any vertex of $\Sigma_n^{(s)}$ is a shedding vertex, and the statement holds when Δ is a simplex.

If Δ is not a simplex, then it has some shedding vertex v such that every $\sigma \in \text{link}_{\Delta} v$ has an x with $\sigma \cup \{x\}$ a face of $\Delta \setminus v$, and in particular every $\sigma \in (\text{link}_{\Delta} v)^{(s-1)}$ has an x such that $\sigma \cup \{x\}$ is a face of $\Delta \setminus v$. Since $\dim(\sigma \cup \{x\}) \leq s$, we get that $\sigma \cup \{x\} \in (\Delta \setminus v)^{(s)}$, so that v is a shedding vertex in $\Delta^{(s)}$. By induction, $\Delta^{(s)}$ is vertex decomposable. \square

4. CHORDAL CLUTTERS

Before introducing our definition of a chordal clutter, we recall the definition and main structure theorem of a chordal graph. A graph is *chordal* if every induced cyclic subgraph of G has length 3. A vertex v of G is *simplicial* if the neighborhood of v in G is a complete subgraph. The main theorem characterizing chordal graphs is:

Theorem 4.1. (essentially G. Dirac [9]) *A graph G is chordal if and only if every induced subgraph of G has a simplicial vertex.*

Most of the attempts at extending the definition of chordal to clutters in algebraic combinatorics have centered around extending the definition of simplicial vertex, and ours will be no exception.

Definition 4.2. Let \mathcal{C} be a clutter. A vertex v of \mathcal{C} is *simplicial* if for every two circuits e_1 and e_2 of \mathcal{C} that contain v , there is a third circuit e_3 such that $e_3 \subseteq (e_1 \cup e_2) \setminus \{v\}$.

In the case where G is a graph, this is obviously the usual definition of a simplicial vertex.

Remark 4.3. Compare Definition 4.2 with Example 2.1! A vertex v is simplicial exactly when the weak circuit exchange property of matroids is satisfied for all edges containing v .

Definition 4.4. A clutter \mathcal{C} is *chordal* if every minor of \mathcal{C} has a simplicial vertex.

Example 4.5. Some chordal clutters:

- (1) Chordal graphs: If G is a graph, then G/v is (up to singleton circuits) the induced subgraph $G \setminus N[v]$. Hence the definition of chordal clutter specializes in graphs to the usual definition of chordal.
- (2) The *complete d -uniform clutter* \mathcal{K}_n^d is the clutter with n vertices and circuit set $\binom{V}{d}$. Since $\mathcal{K}_n^d/v \cong \mathcal{K}_{n-1}^{d-1}$, $\mathcal{K}_n^d \setminus v \cong \mathcal{K}_{n-1}^d$, and every vertex is simplicial, the complete d -uniform clutter is chordal.
- (3) Matroid circuits: Since every vertex of a matroid circuit clutter is simplicial, and every deletion or contraction of a matroid is another matroid, we see that any matroid circuit clutter is chordal.
E.g., \mathcal{K}_n^d is the circuit clutter of a uniform matroid.

Example 4.6. Van Tuyl and Villarreal [28] define a clutter \mathcal{C} to have the *free vertex property* if every minor of \mathcal{C} has a *free vertex*, that is, a vertex appearing in exactly one circuit of \mathcal{C} . We observe that a free vertex is simplicial, so clutters with the free vertex property are chordal. Clutters with the free vertex property were shown to be shellable in [28, Theorem 5.3], and a restricted case of Proposition 5.2 for free vertices was shown in [23, Theorem 2.8].

Van Tuyl and Villarreal notice [28, Corollary 5.7] that if G is a chordal graph, then the clutter \mathcal{C} with vertices $V(G)$ and circuits consisting of all cliques in G has the free vertex property. The Graham-Yu-Özsoyglu algorithm from database theory can be used to show that every clutter with the free vertex property has this form: a helpful reference is [1, especially Theorem 3.4]. Specifically, the Graham-Yu-Özsoyglu algorithm chooses a free vertex v contained in a unique circuit

e , and deletes v if $e \setminus v$ is strictly contained in another circuit, and contracts v otherwise – the algorithm terminates if and only if \mathcal{C} is the clutter of cliques of a chordal graph.

Remark 4.7. Clutters (and more generally hypergraphs) which have the free vertex property have often been referred to as “acyclic”. Since despite the name these clutters may have cycles, we prefer the free vertex property terminology.

Example 4.8. Emtander [13], extending ideas from Hà and Van Tuyl [19], has a different but related definition of chordal for d -uniform clutters. Let a vertex v be a *complete-neighborhood vertex* if the induced subclutter on $S = \{y : x, v \in e\}$ is the complete d -uniform clutter, i.e. has circuits $\binom{S}{d}$. Emtander calls a d -uniform clutter “chordal” if every induced subclutter either has a complete-neighborhood vertex, or else no circuits.

A complete-neighborhood vertex is clearly simplicial in our sense, but Emtander requires only deletions to have simplicial vertices, while we require both deletions and contractions. Examples which are chordal in our sense but not in Emtander’s are easy to come by (most matroids will do). An example which has complete-neighborhood vertices in every induced subclutter but is not chordal is the clutter \mathcal{C} with circuits $\{1, 2, 3\}, \{3, 4, 5\}, \{5, 6, 7\}, \{7, 8, 1\}$. Every induced subclutter of \mathcal{C} has the free vertex property, but contracting 2, 4, 6, and 8 leaves the cyclic graph C_4 . (It follows immediately that $I(\mathcal{C})$ is not shellable or sequentially Cohen-Macaulay.)

As Emtander’s primary interest for clutters with complete-neighborhood vertices was in linear resolutions (as was that of Hà and Van Tuyl), we discuss this further in Section 6.

Example 4.9. The clutter with circuits $\{1, 2, 3\}, \{1, 4, 5\}, \{2, 3, 4, 5\}$ has simplicial vertex 1, and is easily verified to be chordal; but is not a chordal graph or matroid circuit clutter, and does not have the free vertex property.

While chordal graphs are essentially defined in terms of forbidden minors (i.e., the cyclic graphs of length ≥ 4), a forbidden minor characterization of chordal clutters is not known. We will give some partial results for this interesting question in Section 7, but we first present more conclusive positive results in Sections 5 and 6.

5. SHELLABILITY OF THE INDEPENDENCE COMPLEX

Our main goal of this section will be to prove Theorem 1.1.

Recall [30, Lemma 6] that if G is a graph with vertices v and w such that $N[v] \subseteq N[w]$, then w is a shedding vertex in $I(G)$. Motivated by this result, we define a *neighborhood containment pair* of a clutter \mathcal{C} to be a vertex v and a circuit e with $v \in e$ such that if $v \in e_2$ for any circuit $e_2 \neq e$, then there exists a circuit $e_3 \subseteq (e \cup e_2) \setminus v$. Thus, a simplicial vertex forms a neighborhood containment pair with any circuit containing it.

Lemma 5.1. *If (v, e) is a neighborhood containment pair in \mathcal{C} then $\sigma = e \setminus \{v\}$ is a shedding face of $I(\mathcal{C})$.*

Proof. Suppose that τ is a face of $I(\mathcal{C})$ containing σ . Then $\tau \cup v$ contains e , so is not a face (and in particular $v \notin \tau$). For $x \in \sigma$, if $(\tau \setminus x) \cup v$ is not a face of $I(\mathcal{C})$ (hence of $I(\mathcal{C}) \setminus \sigma$), then $(\tau \setminus x) \cup v$ contains some circuit e_2 with $v \in e_2$. But then the neighborhood containment condition gives an $e_3 \subseteq (e \cup e_2) \setminus v \subseteq \tau$, contradicting the choice of τ as a face. Hence any such x can be exchanged for v , fulfilling the shedding face exchange axiom. \square

Proposition 5.2. *If \mathcal{C} is a clutter containing a simplicial vertex v , and if every proper contraction of \mathcal{C} is shellable, then \mathcal{C} is shellable.*

Proof. Let e_1, \dots, e_k be the circuits containing v , and $\sigma_1, \dots, \sigma_k$ be the associated shedding faces $e_i \setminus \{v\}$. Let $\mathcal{C}_0 = \mathcal{C}$, and \mathcal{C}_i be generated by the minimal sets of $\mathcal{C} \cup \{\sigma_1, \dots, \sigma_i\}$, so that $I(\mathcal{C}_i) = I(\mathcal{C}) \setminus \sigma_1 \setminus \dots \setminus \sigma_i$.

Then since there is an $e' \subseteq (e_i \cup e_j) \setminus v = \sigma_i \cup \sigma_j$, we get that \mathcal{C}_{i-1} has some $e'' \subseteq e' \subseteq \sigma_i \cup \sigma_j$. In $\mathcal{C}_{i-1}/\sigma_i$ the circuit e'' contracts to $e''' \subseteq \sigma_j$. In particular the minimal sets of $\mathcal{C}_{i-1}/\sigma_i$ are the same as those of \mathcal{C}/σ_i . We have shown that $\mathcal{C}_{i-1}/\sigma_i = \mathcal{C}/\sigma_i$.

It is straightforward to check that v is simplicial in each of $\mathcal{C}_1, \dots, \mathcal{C}_{k-1}$, and thus σ_i is a shedding face in $I(\mathcal{C}_{i-1})$ by Lemma 5.1. Every required link of the form $\text{link}_{I(\mathcal{C}_{i-1})} \sigma_i = I(\mathcal{C}_{i-1}/\sigma_i) = I(\mathcal{C}/\sigma_i)$ is shellable. The vertex v is isolated in $I(\mathcal{C}_k)$, so that $I(\mathcal{C}_k) = I(\mathcal{C}/v) * v$ is shellable; while each $I(\mathcal{C}_i)$ is shellable by an inductive argument on k , using Lemma 3.4. \square

An intuitive explanation of the proof of Proposition 5.2 is that deleting each $\sigma_i = e_i \setminus \{v\}$ from $I(\mathcal{C})$ leaves $(\mathcal{C}/v) \dot{\cup} \{v\}$ as the minimal non-face clutter.

Corollary 5.3. *Let \mathcal{C} be a clutter with maximal circuit cardinality k , such that every contraction of \mathcal{C} has a simplicial vertex. Then $I(\mathcal{C})$ is $(k-2)$ -decomposable, hence shellable and sequentially Cohen-Macaulay.*

Proof. By induction and noting that each shedding face produced in Proposition 5.2 has dimension at most $k-2$. \square

We thus have the following specialization of Theorem 1.1:

Corollary 5.4. *If \mathcal{C} is a chordal clutter with maximal circuit cardinality k , then $I(\mathcal{C})$ is $(k-2)$ -decomposable, hence shellable and sequentially Cohen-Macaulay.*

We also break out the statement of Corollary 5.3 in the case where \mathcal{C} is a graph.

Corollary 5.5. *If G is a graph such that $G \setminus N[A]$ has a simplicial vertex for any independent set A , then G is vertex decomposable.*

The family of graphs given in Corollary 5.5 is a considerably more general family than that of chordal graphs, including for example simplicial graphs [7], the family of graphs considered in [15, Theorem 3.2], etc. The obvious question suggested is:

Problem 5.6. Classify the forbidden subgraphs of the family of graphs from Corollary 5.5. More generally, classify the forbidden minors of the family of clutters from Corollary 5.3.

6. LINEAR RESOLUTIONS AND ALEXANDER DUALITY

Recall that the complement \bar{G} of a graph G is the graph with the same vertex set and circuit set $\{xy : xy \notin E(G)\}$. There is a similar definition in d -uniform clutters for any d . An important theorem in the algebraic combinatorics of a chordal graph is:

Theorem 6.1. (Fröberg [17]) *Let G be a graph. Then the circuit ideal of G has a linear resolution if and only if \bar{G} is chordal.*

In this section we generalize the “if” direction of Theorem 6.1 from chordal graphs to chordal clutters. We will first need to recall some facts about Alexander duality.

The *Alexander dual* of a simplicial complex Δ (denoted Δ^\vee) is the simplicial complex with vertices $V = V(\Delta)$ and facets

$$\{V \setminus e : e \text{ a circuit of } \mathcal{C}(\Delta)\}.$$

The vertex set that we consider Δ over has unusually great importance in this definition, and if we wish to emphasize the vertex set that we are operating over we will use a subscript, e.g. Δ_V^\vee .

Example 6.2. $I(K_n)^\vee$ is the $(n-3)$ -skeleton of an $(n-1)$ -simplex.

Example 6.3. Let Δ be the n -simplex. Then $\Delta^\vee = \{\}$, the (-1) -simplex, and $(\partial\Delta)^\vee = \{\emptyset\}$, the (-1) -sphere.

Example 6.4. Let Δ be any simplicial complex, with vertex set V . Expanding on our remark above about the vertex set, let us consider $\Delta_{V \cup \{x\}}^\vee$ for $x \notin V$. Since $\{x\}$ is a minimal non-face, V is a facet of $\Delta_{V \cup \{x\}}^\vee$; the other facets are those of the cone over x of Δ^\vee .

The Alexander dual allows us to reduce the question of the existence of a linear resolution to topological combinatorics:

Theorem 6.5. (Eagon and Reiner [11, Theorem 3]) *Let Δ be a simplicial complex. The circuit ideal of Δ has a linear resolution if and only if Δ^\vee is Cohen-Macaulay.*

In particular, one approach to proving the “if” direction of Theorem 6.1 is as follows:

Theorem 6.6. (Eagon and Reiner [11, Proposition 8]) *If G is a chordal graph, then $I(\overline{G})^\vee$ is vertex decomposable.*

6.1. Review of Alexander duality. In order to extend Theorem 6.6 to chordal clutters, we will need some facts about Alexander duality. We refer the reader also to [22, Section 6].

Lemma 6.7. *If Δ is any simplicial complex on vertex set V then*

- (1) $\tilde{H}_i(\Delta) \cong \tilde{H}^{|V|-i-3}(\Delta^\vee)$.
- (2) $(\Delta^\vee)^\vee = \Delta$.
- (3) $(\Delta \setminus v)^\vee = \text{link}_{\Delta^\vee} v$ and $(\text{link}_\Delta v)^\vee = \Delta^\vee \setminus v$.
- (4) Δ is pure of dimension d if and only if $\mathcal{C}(\Delta^\vee)$ is $(|V| - d - 1)$ -uniform.

Proof. (1) Apply the topological Alexander Duality Theorem, considering the boundary of the n -simplex as the union of a Δ and (a space homotopic to) Δ^\vee . See [2] for details, as well as another, combinatorial proof.

(2) The faces of Δ^\vee are all σ such that $V \setminus \sigma$ is not a face of Δ , so a minimal non-face of Δ^\vee is the complement of a facet of Δ ; the result then follows by definition.

(3) Since $\mathcal{C}(\Delta \setminus v) = \mathcal{C}(\Delta) \setminus v$, the faces of $(\text{link}_\Delta v)^\vee$ are all complements in $V \setminus \{v\}$ of circuits in $\mathcal{C}(\Delta)$ not containing v , which is exactly $\text{link}_{\Delta^\vee} v$. The other statement follows by duality.

(4) Immediate from definition and duality. □

Remark 6.8. The Alexander dual has been studied in topological combinatorics at least as far back as [22, Section 6]. It has also been studied in the context of combinatorial optimization under the name *blocker* or *transversal*, and it is in this context that Lemma 6.7 parts (2) and (3)

were first observed. We refer the reader to [8] for further background and references from the combinatorial optimization point of view.

6.2. Cohen-Macaulay Alexander duals. If \mathcal{C} is a clutter, then define $c_d(\mathcal{C})$ to be the clutter with the same vertex set V as \mathcal{C} and circuit set

$$\{e \subseteq V : |e| = d, e \text{ not a circuit of } \mathcal{C}\}.$$

In the special case that \mathcal{C} is d -uniform, this is the *complement* of \mathcal{C} discussed in [19, 13]. We refer to the circuits of $c_d(\mathcal{C})$ as d -non-circuits of \mathcal{C} .

We start with a lemma relating contraction in \mathcal{C} with contraction in $c_d(\mathcal{C})$:

Lemma 6.9. *Let \mathcal{C} be a clutter with no circuits of cardinality $(d - 1)$, and v be a simplicial vertex. Then $c_d(\mathcal{C})/v = c_{d-1}(\mathcal{C}/v)$.*

Proof. Suppose that e is a d -non-circuit of \mathcal{C} with $v \notin e$, and that e is the only such non-circuit contained in the set $e \cup v$. Then the induced subclutter of \mathcal{C} on the set $e \cup \{v\}$ is a complete clutter with one circuit removed ($\mathcal{K}_{d+1}^d \setminus \{e\}$), which contradicts the hypothesis that v is simplicial. It follows that every d -non-circuit of $c_d(\mathcal{C})$ contains a $(d - 1)$ -set e' which is a circuit of $c_d(\mathcal{C})/v$, such that $e' \cup \{v\}$ is a non-circuit of \mathcal{C} . Thus such e' are precisely the circuits of $c_d(\mathcal{C})/v$.

Because there are no circuits with $d - 1$ vertices in \mathcal{C} , the $(d - 1)$ -circuits of \mathcal{C}/v are exactly the e such that $e \cup \{v\}$ is a circuit of \mathcal{C} . We have that

$$\{e : |e| = d - 1, e \cup \{v\} \text{ a non-circuit of } \mathcal{C}\} = c_{d-1}(\mathcal{C}/v) = c_d(\mathcal{C})/v. \quad \square$$

Notice that \mathcal{C}/v is in general not a uniform clutter, even if the starting clutter \mathcal{C} was uniform. It is for this reason that we work with c_d , which is always d -uniform, rather than with a more straightforward complement of d -uniform clutters.

Lemma 6.10. *If v is a simplicial vertex of \mathcal{C} such that $\mathcal{C} \setminus v$ has at least one d -non-circuit ($d \geq 2$), then v is a shedding vertex in $I(c_d(\mathcal{C}))^\vee$.*

Proof. Suppose that σ is a facet of $\text{link}_{I(c_d(\mathcal{C}))^\vee} v$, so that $\sigma = (V \setminus \{v\}) \setminus e$ for some d -non-circuit e not containing v . (Such a facet exists by the condition requiring $\mathcal{C} \setminus v$ to have at least one d -non-circuit.) Since $d \geq 2$ there are vertices $w_1, w_2 \in e$, and we let e_i be the set $(e \setminus w_i) \cup \{v\}$.

If both e_1 and e_2 are circuits of \mathcal{C} , then $(e_1 \cup e_2) \setminus v = e$ is also a circuit, a contradiction; so at least one e_i is a d -non-circuit. But then $\tau = V \setminus e_i$ is a facet of $I(c_d(\mathcal{C}))^\vee \setminus \{v\}$ with $\tau = \sigma \cup \{w_i\}$, meeting the requirement for a shedding vertex. \square

We are now ready to prove:

Theorem 6.11. *If \mathcal{C} is a chordal clutter with minimal circuit cardinality d , then $I(c_d(\mathcal{C}))^\vee$ is vertex decomposable.*

Proof. We prove by induction, with base cases as follows: If $c_d(\mathcal{C})$ has no circuits, then $I(c_d(\mathcal{C}))^\vee$ is the degenerate complex $\{\}$, which we defined to be (-1) -decomposable. If $d = 1$ and there is a circuit in $c_d(\mathcal{C})$, then $I(c_d(\mathcal{C}))^\vee$ is a collection of codimension 1 faces of a simplex, which Eagon and Reiner observe to be vertex decomposable [11, proof of Proposition 8].

Let v be a simplicial vertex of \mathcal{C} . Then

$$\text{link}_{I(c_d(\mathcal{C}))^\vee} v = I(c_d(\mathcal{C}) \setminus v)^\vee = I(c_d(\mathcal{C} \setminus v))^\vee$$

is vertex decomposable by induction, and

$$I(c_d(\mathcal{C}))^\vee \setminus v = I(c_d(\mathcal{C})/v)^\vee = I(c_{d-1}(\mathcal{C}/v))^\vee$$

is vertex decomposable by induction and Lemma 6.9.

If $\mathcal{C} \setminus v$ has a d -non-circuit, then v is a shedding vertex by Lemma 6.10, hence $I(c_d(\mathcal{C}))^\vee$ is vertex decomposable. Otherwise, v is contained in every circuit of $c_d(\mathcal{C})$, hence in no facet of $I(c_d(\mathcal{C}))^\vee$, so that $I(c_d(\mathcal{C}))^\vee = I(c_d(\mathcal{C}))^\vee \setminus v$, which is vertex decomposable by induction. \square

We have proved the following generalization of Theorem 1.2.

Corollary 6.12. *If \mathcal{C} is a chordal clutter with minimal circuit cardinality d , then the circuit ideal of $c_d(\mathcal{C})$ has a linear resolution.*

As mentioned in Example 4.8, there are clutters such that every subclutter contains a complete-neighborhood vertex, but that are not chordal. We can however use a similar technique to show that clutters with a complete-neighborhood vertex in every induced subclutter are vertex decomposable, improving the previous result [14, Theorem 4.3] that such clutters are shellable:

Proposition 6.13. *Let \mathcal{C} be a d -uniform clutter such that every induced subclutter has a complete-neighborhood vertex. Then $I(c_d(\mathcal{C}))^\vee$ is vertex decomposable.*

Proof. By induction we may assume that $\text{link}_{I(c_d(\mathcal{C}))^\vee} v = I(c_d(\mathcal{C} \setminus v))^\vee$ is shellable. A complete-neighborhood vertex v is simplicial, thus either v is a shedding vertex or else $I(c_d(\mathcal{C}))^\vee = I(c_d(\mathcal{C}))^\vee \setminus v$ as simplicial complexes, exactly as in the proof of Theorem 6.11. It remains only to show that $I(c_d(\mathcal{C})/v)^\vee$ is shellable.

Let $N = \bigcup_{v \in e} (e \setminus \{v\})$ be the neighborhood of v . The induced subclutter on N is $\mathcal{K}_{|N|}^d$. By Lemma 6.9, $I(c_d(\mathcal{C})/v)^\vee$ has circuits $\{e :$

$|e| = d - 1$, $e \cup \{v\}$ a non-circuit of \mathcal{C} , that is, all e of cardinality $d - 1$ such that $e \not\subseteq N$.

It follows that $I(c_d(\mathcal{C})/v)^\vee$ is the pure $|V| - d - 2$ skeleton of the complex Δ on $V \setminus \{v\}$ with the single non-face $V \setminus (\{v\} \cup N)$. Then $\mathcal{C}(\Delta)$ is the disjoint union of a single $(|V| - |N| - 1)$ -edge with $|N|$ disconnected vertices, hence Δ is the join of a simplex and a simplex boundary. It follows by vertex decomposability of the simplex, by Proposition 3.9 and by Lemma 3.12 that $I(c_d(\mathcal{C})/v)^\vee$ is vertex decomposable. \square

Corollary 6.14. (Emtander [13, Theorem 4.1]) *Let \mathcal{C} be a d -uniform clutter such that every induced subclutter has a complete-neighborhood vertex. Then the circuit ideal of $c_d(\mathcal{C})$ has a linear resolution.*

Corollary 6.12 and Corollary 6.14 extend one direction of Theorem 6.1 to d -uniform clutters. One might ask if there is an extension of the other direction as well, i.e., a characterization of d -uniform clutters \mathcal{C} such that the circuit ideal of \mathcal{C} has a linear resolution. We point out that a classification of such d -uniform clutters classifies the simplicial complexes of dimension $|V| - d - 1$ which are Cohen-Macaulay, so that characterizing such clutters for all d gives a characterization of all Cohen-Macaulay simplicial complexes. As Herzog, Hibi, and Zheng [21] observe in the context of classifying all Cohen-Macaulay graphs, this is a very difficult problem, and unlikely to have a simple answer.

Classifying the clutters in some family that have a circuit ideal with linear resolution is still an interesting and often approachable problem. Theorem 6.1 is one example of such a classification. We comment that Theorem 6.1 is possible because the Alexander dual is of very high dimension (with respect to $|V|$). Another classification result is: if \mathcal{C} is a $(|V| - 2)$ -uniform clutter, then obviously the circuit ideal of \mathcal{C} has linear resolution if and only if $I(\mathcal{C})^\vee$ is a connected 1-complex. In this example, the Alexander dual has low dimension.

Problem 6.15. Classify all 3-uniform clutters such that the circuit ideal has a linear resolution. Are all such clutters shellable?

Our complement operation c_d outputs a d -uniform clutter, even if the starting clutter was not uniform. If \mathcal{C} is not d -uniform, then $I(\mathcal{C})^\vee$ is not pure, hence not Cohen-Macaulay; but it could still be sequentially Cohen-Macaulay. Algebraically, this corresponds with the circuit ideal being *component-wise linear* [20].

Problem 6.16. Find an interesting extension of Theorem 6.11 involving non-uniform clutters and component-wise linear circuit ideals.

7. FORBIDDEN MINORS

7.1. Obstructions to shellability. Wachs defined an *obstruction to shellability* to be a non-shellable simplicial complex such that every induced subcomplex is shellable. The obstructions to shellability that are *flag complexes* (independence complexes of graphs) were recently classified: Francisco and Van Tuyl [16] showed that chordal graphs are sequentially Cohen-Macaulay and that the n -cycle is an obstruction to shellability for $n \neq 3, 5$. The author [30] showed that every complex containing no such cycle is shellable. We see a close relationship between the obstructions to shellability in flag complexes and the forbidden subgraphs of a chordal graph.

It is easier to study obstructions to shellability in the special case of flag complexes for at least two reasons. The first is that graphs are better studied than clutters, and so there were pre-existing theorems relating the forbidden subgraphs characterization of chordal graphs to the simplicial vertex characterization. The second is that every link in a flag complex can be expressed as an induced subgraph: $\text{link}_{I(G)} v = I(G \setminus N[v])$. We try to partially remedy the latter with the following alternate definition:

Definition 7.1. A complex Δ is a *dc-obstruction to shellability* (where *dc* stands for “deletion-contraction”) if Δ is non-shellable, but both every induced subcomplex and every link are shellable. A non-shellable complex Δ such that $\text{link}_{\Delta} v$ is shellable for every $v \in V(\Delta)$ is a *c-obstruction to shellability*, and an obstruction to shellability in the sense of Wachs is a *d-obstruction to shellability*.

Example 7.2. The complex Δ with facets $\{\{1, 2, 3\}, \{3, 4, 5\}, \{1, 5\}\}$ is a *d-obstruction* but not a *dc-obstruction* to shellability, since deleting any vertex leaves a connected complex with a single 2-face, but $\text{link}_{\Delta} 3$ is two disconnected circuits, hence not shellable. Similarly for the family constructed in [29, Proposition 1]

Since *d-obstructions* to shellability allow the possibility of complexes where non-shellability is controlled by a proper (non-induced) subcomplex, we regard the definition of *dc-obstructions* to shellability as the more natural definition, although both are interesting. Since every link in a shellable complex is shellable, a classification of *c-obstructions* to shellability would give a classification of all shellable complexes, which seems unlikely to be achieved.

Wachs conjectured [29] that there are a finite number of k -dimensional obstructions to shellability for any fixed k . We suggest the weaker conjecture:

Conjecture 7.3. *There are a finite number of k -dimensional dc -obstructions to shellability for any fixed k .*

7.2. Examples of forbidden minors. A *forbidden subclutter* of some family \mathcal{F} of clutters is a clutter \mathcal{C} not in \mathcal{F} such that every induced subclutter is in \mathcal{F} . A *forbidden minor* of some family \mathcal{F} of clutters is a clutter \mathcal{C} that is not in \mathcal{F} , but such that every minor (obtained by both deletion and contraction) is in \mathcal{F} . Every forbidden minor is a forbidden subclutter, but the converse is not true. We remark that every nonshellable forbidden subclutter to chordality is a d -obstruction to shellability, while every nonshellable forbidden minor to chordality is a dc -obstruction to shellability (since these two families are the forbidden subclutters and forbidden minors to the family of clutters with every subclutter shellable). Since the family of forbidden minors to chordality is smaller, classifying forbidden minors is likely easier than classifying forbidden subclutters.

We present several examples of infinite families of forbidden minors.

Example 7.4. Since chordal graphs are a special case of chordal clutters, the n -cycle graph C_n is a forbidden minor for $n \geq 4$. We recall [16, Proposition 4.1] that the independence complexes of the cyclic graphs are non-shellable except for $I(C_5)$.

Example 7.5. Let \mathcal{Z}_n^k be the clutter with vertex set \mathbb{Z}_n and circuits consisting of every k consecutive elements. Thus, $\mathcal{Z}_n^2 \cong C_n$, and more generally \mathcal{Z}_n^k are the obvious k -uniform extension of the cyclic graphs. Any vertex (hence every vertex) of \mathcal{Z}_n^k is simplicial if and only if $k = n$ or $n - 1$, so \mathcal{Z}_n^k is chordal if and only if $k = n$ or $n - 1$. Deleting any vertex leaves a clutter with the free vertex property, so \mathcal{Z}_n^k ($k \neq n, n - 1$) are forbidden subclutters to chordality. In some cases, for example \mathcal{Z}_5^3 , they may also be forbidden minors.

We take a brief detour to discuss some cases when \mathcal{Z}_n^k is not a forbidden minor to chordality, i.e., when \mathcal{Z}_n^k has a non-chordal contraction.

Lemma 7.6. *If $\ell k \leq n \leq \ell(k+1)$ and $k > 2$, then \mathcal{Z}_n^k has a contraction isomorphic to $\mathcal{Z}_{n-\ell}^{k-1}$.*

Proof. The condition allows us to pick a set $S = \{v_1, \dots, v_\ell\}$ vertices from $\mathbb{Z}_n = V(\mathcal{Z}_n^k)$ so that every 2 vertices in S have k or $k - 1$ vertices between them. Contracting S is easily seen to give the desired minor. \square

E.g., \mathcal{Z}_6^3 is not a forbidden minor to chordality, since it contains a contraction minor isomorphic to the cyclic graph C_4 ($= \mathcal{Z}_4^2$).

More broadly, we could consider “clutters of cyclic type”: clutters on vertex set Z_n with all circuits consisting of consecutive elements (possibly of different cardinalities). The next example, however, shows that not all forbidden minors have this form; moreover, the results of Section 7.3 seem to say that such a form is relatively uncommon.

Example 7.7. Let \mathcal{X}_n be the clutter with vertex set $[2n]$ and circuits $\{\text{odd vertices}\}$, $\{\text{even vertices}\}$, and $\{i, i+1\}$ for all odd i . By symmetry no vertex is simplicial, and deleting or contracting any vertex leaves the same clutter (up to isomorphism). Any such deletion or contraction removes one of the two circuits with cardinality n , leaving a clutter with the free vertex property. Thus \mathcal{X}_n is a forbidden minor to chordality.

The independence complex $I(\mathcal{X}_n)$ is the boundary complex of the n -dimensional cross-polytope with two opposing facets removed, a non-shellable complex. Hence $I(\mathcal{X}_n)$ is an dc -obstruction to shellability.

We note that of the dc -obstructions to shellability M_5 , M_6 , and M_7 considered by Wachs, we have $M_5 \cong I(\mathcal{Z}_5^3)$, $M_6 \cong I(\mathcal{X}_3)$, and $M_7 \cong I(C_7)$.

Question 7.8. *What are the forbidden subclutters or minors to chordality, and what are their relationship with obstructions to shellability?*

7.3. Computational results. Computation with GAP [18] yields exactly two forbidden minors to chordality on 5 vertices: the cyclic graph C_5 and the clutter \mathcal{Z}_5^3 discussed in Example 7.5. Both have homotopy type S^1 . \mathcal{Z}_5^3 is a dc -obstruction to shellability, while $I(C_5)$ is shellable.

On 6 vertices, a similar computation yields 294 (isomorphism classes of) forbidden minors to chordality on 6 vertices. There are an additional 96 clutters containing a C_5 minor (but no other non-chordal minor), all 96 of which are shellable. Of the 294 forbidden minors to chordality, 273 are shellable and 21 are not – a complete list together with source code is available on my web page, currently at <http://www.math.wustl.edu/~russw>. The computation took several hours on a 2.4 Ghz MacBook.

The 21 non-shellable forbidden minors to chordality are the dc -obstructions to shellability on 6 vertices. These clutters and their independence complexes are summarized in Table 1. The clutters and simplicial complexes are written in compact notation, so that, for example, 12 represents the set $\{1, 2\}$. The fourth column of Table 1 represents the homotopy type of the pure top dimensional skeleton, as computed by automatic collapsing of free faces. Since in every case this is of lower dimension than the top dimensional face, this suffices to show that none of these complexes is sequentially Cohen-Macaulay.

TABLE 1. dc -obstructions to shellability on 6 vertices

	Clutter of minimal non-faces	Independence complex	Top skel.
1.	12, 13, 24, 35, 46, 56	145, 16, 236, 25, 34	S^0
2.	12, 13, 14, 235, 236, 245, 246, 256, 345, 346, 356, 456	156, 234, 25, 26, 35, 36, 45, 46	S^0
3.	12, 13, 14, 235, 236, 245, 256, 345, 356, 46	156, 234, 25, 26, 35, 36, 45	S^0
4.	12, 13, 14, 235, 236, 256, 356, 45, 46	156, 234, 25, 26, 35, 36	S^0
5.	12, 13, 14, 235, 246, 256, 36, 45	156, 234, 25, 26, 35, 46	S^0
6.	12, 13, 145, 146, 235, 236, 245, 246, 256, 345, 346, 356, 456	14, 156, 234, 25, 26, 35, 36, 45, 46	S^0
7.	12, 13, 145, 146, 235, 245, 246, 256, 345, 36, 456	14, 156, 234, 25, 26, 35, 45, 46	S^0
8.	12, 13, 145, 146, 245, 26, 346, 35, 456	14, 156, 234, 25, 36, 45, 46	S^0
9.	12, 13, 145, 234, 236, 245, 246, 345, 346, 56	146, 15, 235, 24, 26, 34, 36, 45	S^0
10.	12, 13, 145, 236, 24, 345, 346, 56	146, 15, 235, 26, 34, 36, 45	S^0
11.	12, 13, 145, 24, 345, 36, 56	146, 15, 235, 26, 34, 45	S^0
12.	12, 13, 234, 235, 245, 345, 46, 56	145, 16, 236, 24, 25, 34, 35	S^0
13.	12, 134, 135, 136, 145, 146, 235, 236, 245, 246, 256, 345, 346, 356, 456	13, 14, 156, 234, 25, 26, 35, 36, 45, 46	S^0
14.	12, 134, 135, 136, 145, 234, 236, 245, 246, 345, 346, 56	13, 146, 15, 235, 24, 26, 34, 36, 45	S^0
15.	12, 134, 135, 146, 235, 246, 256, 36, 45	13, 14, 156, 234, 25, 26, 35, 46	S^0
16.	123, 124, 125, 126, 134, 135, 136, 145, 146, 235, 236, 245, 246, 256, 345, 346, 356, 456	12, 13, 14, 156, 234, 25, 26, 35, 36, 45, 46	S^0
17.	12, 134, 135, 146, 235, 246, 256, 345, 346, 356, 456	136, 145, 156, 234, 236, 245, 35, 46	S^1
18.	12, 134, 135, 234, 246, 345, 346, 56	136, 145, 146, 235, 236, 245, 34	S^1
19.	12, 134, 256, 35, 46	136, 145, 156, 234, 236, 245	S^1
20.	123, 124, 125, 126, 134, 135, 146, 235, 246, 256, 345, 346, 356, 456	12, 136, 145, 156, 234, 236, 245, 35, 46	S^1
21.	1234, 1235, 1246, 1356, 2456, 3456	1236, 1245, 1256, 1345, 1346, 1456, 2345, 2346, 2356	S^2

We notice that the first 16 rows of the table represent simplicial complexes which consist of two disjoint 2-dimensional faces, with enough edges between them to prevent a non-shellable minor. Line 1 represents the cyclic graph C_6 and its independence complex. Lines 17 to 20 represent simplicial complexes consisting of an annulus formed by six 2-facets, together with some additional 1-dimensional facets. Line 19 is isomorphic to the clutter \mathcal{X}_3 discussed in Example 7.7. Line 21 is isomorphic to the clutter \mathcal{Z}_6^4 discussed in Example 7.5, via the ordering of vertices 1, 2, 4, 6, 5, 3. Lines 1 and 21 are the only clutters of cyclic type.

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