Please complete the below statement.

I have completed my work on the below exam within 2 hours. I have not made use of any books, notes, calculators, or internet resources; and have not discussed the problems with anyone.

Signature: ______________________________

Name (Printed): ______________________________

Exam started at   and finished at   (times), on   (date).

Math 493

Final Examination – December 19, 2011

Name ________________________________

General Instructions: Please answer the following, showing all your work, and without the aid of books, notes, or calculators. Total number of points = 40. Write all probabilities involving normal random variables in terms of the standard normal pdf.

Integration by parts formula: $\int u\,dv = uv - \int v\,du$.

1. (1 points each) True/False. Please read the statements carefully, as no partial credit will be given.

(a) ____ If X and Y are independent continuous random variables with pdfs f_X and f_Y, then $X+2Y$ has density function

$$f_{X+2Y}(t) = \frac{1}{2}\int_{-\infty}^{\infty} f_X(t-\heartsuit) f_Y(\frac{\heartsuit}{2})\, d\heartsuit.$$

(b) ____ If $Y_1, Y_2, \ldots$ are i.i.d. exponential random variables with $E(Y_i) = \mu$, then

$$\Pr(Y_1 + \cdots + Y_n < n \cdot \mu) \to 0 \quad \text{as } n \to \infty.$$

(c) ____ If $\Pr(X+Y+Z<1) = 0.99$, then $E(X+Y+Z) < 100$.

(d) ____ If A and B are independent events, then the probability that exactly one of the two events occurs is $\Pr(A) + \Pr(B) - 2\Pr(A)\Pr(B)$.

(e) ____ By linearity of variance, we have for any random variables X and Y that

$$\mathrm{Var}(2X+3Y) = 2\,\mathrm{Var}(X) + 3\,\mathrm{Var}(Y).$$

(f) ____ If X and Y are independent random variables, then

$$E(X^2Y^2) = E(X^2)E(Y^2).$$

(g) ____ If X is a Cauchy random variable (with pdf $\frac{1}{\pi(1+x^2)}$), then $E(X) = 0$.

(h) ____ If X and Y are the outcomes of independent die rolls, then $X+Y$ and $X-Y$ are independent random variables.

(i) ____ If X is a continuous random variable with support $[-2, 2]$, then $E(X) = 0$.

2. (a) (6 points) Prove that if X is a continuous uniform random variable on $[a, b]$, then $-X$ is continuous uniform on $[-b, -a]$.

(b) (6 points) Suppose that Z is a standard normal random variable, with pdf $\frac{1}{\sqrt{2\pi}} e^{-t^2/2}$. Find the pdf of Z^2.

3. (a) (6 points) Let X and Y be independent random variables, so that X has the Cauchy distribution (with pdf $\frac{1}{\pi(1+x^2)}$), and Y has the continuous uniform distribution on $[0, 1]$. Find the pdf of $X + Y$.
(It may be helpful to recall that $\frac{d}{dt} \tan^{-1} t = \frac{1}{1+t^2}$).

(b) (5 points) Let X and Y be independent with the exponential pdf

$$f(t) = \begin{cases} e^{-t} & \text{if } t \geq 0 \\ 0 & \text{else.} \end{cases}$$

Find the pdf and expected value of $M = \min(X, Y)$.

4. A standard deck of cards consists of 52 cards, each with a value (2, 3, …, 10, Jack, Queen, King, Ace), and a suite ($\heartsuit, \clubsuit, \diamondsuit, \spadesuit$). You are dealt a hand of 13 cards, uniformly at random.
The random variable Q is calculated as follows. For each suite such that your hand contains the ace and king, count 2. For each suite such that your hand contains the ace but not the king, count 1. For each suite such that your hand does not contain the ace, count 0. (Q is a measure of the strength of your hand in a game such as bridge.)
Thus, for example, if you were dealt the hand

$$\heartsuit AK32 \quad \clubsuit AQ9 \quad \diamondsuit Q8 \quad \spadesuit K742$$

then the value of Q would be 3, since there is an Ace-King in $\heartsuit$ and an Ace in $\clubsuit$.

(a) (4 points) Find $\Pr(Q = 0)$ and $\Pr(Q = 8)$.

(b) (4 points) Calculate $E(Q)$.
(Linearity of expectation may be helpful.)