Math 131
Midterm Examination 3 Solutions – April 27, 2009

General Instructions: You may use a simple calculator that is not graphing or programmable. You may have a 3x5 card, but no other notes.

Part I (70 points): For each of the following 17 problems, mark your answer on the answer card. For Part I, only the answer on the card will be graded.

Problems 1-12: Multiple choice. Each problem is worth 5 points.

1. Calculate the derivative of $\tan^{-1}\left(\frac{x^2}{2}\right)$.

   (a) $\frac{x}{1+x^2}$
   (b) $\frac{x}{1+\frac{x^4}{4}}$
   (c) $\frac{x}{1+x^4}$
   (d) $\frac{1}{1+\frac{x^4}{4}}$
   (e) $\frac{1}{1+x^2}$
   (f) $\frac{x^2}{1+x^2}$
   (g) $\frac{1}{1+\frac{x^4}{4}} \cdot \frac{x^2}{2}$
   (h) The derivative does not exist, because the function is not continuous.

   Differentiating with the chain rule gives the answer to be B.

2. Calculate the derivative of $x \sin^{-1} 3x$.

(a) $\dfrac{1}{\sqrt{1-9x^2}}$

(b) $\dfrac{3}{\sqrt{1-9x^2}}$

(c) $\dfrac{3}{\sqrt{1-9x^2}} + x\sin^{-1}3x$

(d) $\dfrac{3x}{\sqrt{1-9x^2}} + \sin^{-1}3x$

(e) $\dfrac{3x}{\sqrt{1-9x^2}}$

(f) $\dfrac{3x}{\sqrt{1-9x^2}} + x\sin^{-1}3x$

(g) The derivative does not exist, because the function is not continuous.

(h) The derivative does not exist, because $\sin x$ is not an invertible function.

Differentiating with the chain rule and product rule gives the answer to be D.

3. Let $f(x) = \dfrac{(x-1)^2(x-2)^3(x-3)^4}{x^2+1}$. Use logarithmic differentiation to calculate $f'(x)$.

(a) $\left(\dfrac{2}{x-1} + \dfrac{3}{x-2} + \dfrac{4}{(x-3)}\right) \cdot \left(\dfrac{1}{x^2+1}\right) \cdot \left(\dfrac{(x-1)^2(x-2)^3(x-3)^4}{x^2+1}\right)$

(b) $\left(\dfrac{2}{x-1} + \dfrac{3}{x-2} + \dfrac{4}{(x-3)} - \dfrac{2x}{x^2+1}\right)$

(c) $\left(\dfrac{2(x-1)\cdot 3(x-2)^2 \cdot 4(x-3)^3}{2x}\right)$

(d) $\left(\dfrac{2}{x-1} + \dfrac{3}{x-2} + \dfrac{4}{(x-3)} - \dfrac{2x}{x^2+1}\right) \cdot \left(\dfrac{(x-1)^2(x-2)^3(x-3)^4}{x^2+1}\right)$

(e) $\left(\dfrac{2(x-1)\cdot 3(x-2)^2 \cdot 4(x-3)^3}{2x}\right)\left(\dfrac{(x-1)^2(x-2)^3(x-3)^4}{x^2+1}\right)$

(f) $\left(\frac{2}{x-1}+\frac{3}{x-2}+\frac{4}{(x-3)}\right)\cdot\left(\frac{1}{2x}\right)\left(\frac{(x-1)^2(x-2)^3(x-3)^4}{x^2+1}\right)$

(g) $\left(\frac{2}{x-1}+\frac{3}{x-2}+\frac{4}{(x-3)}\right)\cdot\left(\frac{2x}{x^2+1}\right)\cdot\left(\frac{(x-1)^2(x-2)^3(x-3)^4}{x^2+1}\right)$

(h) f is not differentiable.

Let $y=f(x)$. Then

$$\begin{aligned}\ln y &= \ln\frac{(x-1)^2(x-2)^3(x-3)^4}{x^2+1}\\ &= 2\ln(x-1)+3\ln(x-2)+4\ln(x-3)-\ln(x^2+1)\end{aligned}$$

Differentiating both sides implicitly, we get that

$$\frac{y'}{y}=\frac{2}{x-1}+\frac{3}{x-2}+\frac{4}{x-3}-\frac{2x}{x^2+1},$$

and solving for y' gives answer D.

4. Suppose that $f(x)=x\ln x$. The critical points of f are

   (a) 0, 1, and e only
   (b) 0, 1, and e^{-1} only
   (c) 1 only
   (d) e only
   (e) e^{-1} only
   (f) 1 and e^{-1} only
   (g) 0 and e only
   (h) 0 and e^{-1} only

   We calculate $f'(x)$ to be $1\cdot\ln x+\frac{x}{x}=\ln x+1$. This is 0 when $\ln x=-1$, i.e., when $x=e^{-1}$, and never undefined, giving an answer of E.

5. Suppose that $f'(x)=(x-1)(x-2)(x-3)^2$. The intervals where f is increasing are:

(a) $(-\infty, 1)$ only

(b) $(1, 2)$ only

(c) $(2, 3)$ only

(d) $(3, \infty)$ only

(e) $(2, \infty)$ only

(f) $(-\infty, 1)$ and $(2, \infty)$ only

(g) $(-\infty, 1)$ and $(2, 3)$ only

(h) $(1, 2)$ and $(3, \infty)$ only

We check the sign of each term in the derivative:

	$(-\infty, 1)$	1	$(1, 2)$	2	$(2, 3)$	3	$(3, \infty)$
$(x-1)$	-	0	+	+	+	+	+
$(x-2)$	-	-	-	0	+	+	+
$(x-3)^2$	+	+	+	+	+	0	+
$f'(x)$	+	0	-	0	+	0	+

Hence, f is increasing on $(-\infty, 1)$ and $(2, \infty)$ only, answer F.

6. Let $f(x) = x^{4/3} + x^{1/3}$. Find all critical points of f.

(a) $-\frac{1}{4}$ only

(b) 0 only

(c) 1 only

(d) $-\frac{1}{4}$ and 0 only

(e) $-\frac{1}{4}$ and 1 only

(f) 0 and 1 only

(g) $-\frac{1}{4}$, 0, and 1 only

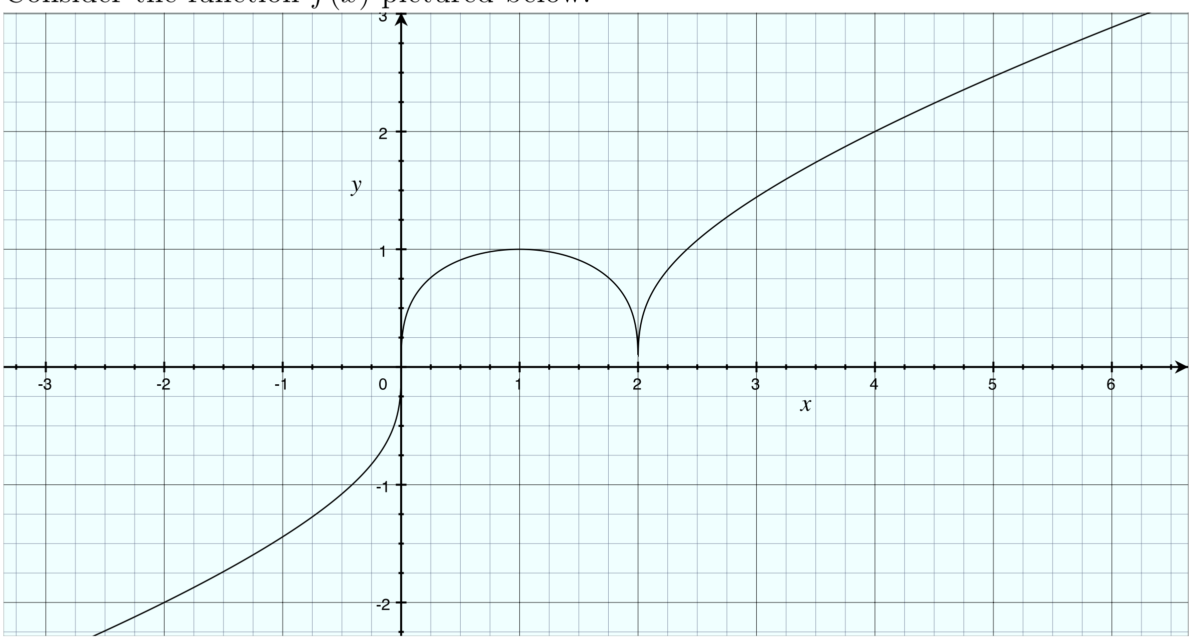
(h) No critical points.

We calculate

$$\begin{aligned} f'(x) &= \frac{4}{3}x^{1/3} + \frac{1}{3}x^{-2/3} \\ &= \frac{1}{3x^{2/3}}(4x+1). \end{aligned}$$

Thus, f' is 0 at $x = -\frac{1}{4}$, and undefined at 0, giving two critical points and answer D.

7. Consider the function $f(x)$ pictured below.



The second derivative of f is positive on the intervals:

(a) $(-\infty, 0)$ only.

(b) $(0, 2)$ only.

(c) $(2, \infty)$ only.

(d) $(-\infty, 2)$ and $(0, 2)$ only.

(e) $(-\infty, 2)$ and $(2, \infty)$ only.

(f) $(0, 2)$ and $(2, \infty)$ only.

(g) $(-\infty, 2)$, $(0, 2)$, and $(2, \infty)$ only.

(h) Everywhere.

(i) Nowhere.

The function is concave up only on $(-\infty, 0)$, answer A.

8. Let $f(x) = x^3 - 6x^2 - 15x$ on the interval $[-6, 6]$. Find the absolute maximum and minimum values of $y = f(x)$.

   (a) Minimum = -658, Maximum = -90
   (b) Minimum = -658, Maximum = 8
   (c) Minimum = -342, Maximum = -90
   (d) Minimum = -342, Maximum = 8
   (e) Minimum = -100, Maximum = 8
   (f) Minimum = -100, Maximum = -90
   (g) Minimum = -100, Maximum = 32
   (h) Minimum = -90, Maximum = 8

   We calculate

$$\begin{aligned} f'(x) &= 3x^2 - 12x - 15 \\ &= 3(x-5)(x+1). \end{aligned}$$

   Thus, f has critical points at 5 and -1. The possible locations of absolute maxes/mins are the endpoints and critical points, i.e., $-6, -1, 5, 6$. We calculate $f(-6) = -342$, $f(-1) = 8$, $f(5) = -100$, and $f(6) = -90$. The largest is 8, the smallest -342, which is answer D.

9. Use linear approximation at $x = 4$ to approximate $\frac{1}{4.01}$.

   (a) $\frac{1}{16} \cdot 4.01$
   (b) $-\frac{1}{16} \cdot 4.01$
   (c) $\frac{1}{16} \cdot 4.01 + \frac{1}{4}$
   (d) $-\frac{1}{16} \cdot 4.01 + \frac{1}{4}$
   (e) $\frac{1}{16} \cdot 4.01 + \frac{1}{2}$
   (f) $-\frac{1}{16} \cdot 4.01 + \frac{1}{2}$

(g) .249376558

(h) 1

We are approximating $\frac{1}{x}$ near $x = 4$. We find the tangent line:

$$y = y(4) + y'(4)(x - 4) = \frac{1}{4} - \frac{1}{4^2}(x - 4)$$

and plug in $x = 4.01$. Simplifying, we get answer F.

10. Let $f(x) = \dfrac{3}{1 + e^{-x}}$. Find all critical points of $f(x)$.

    (a) -1 only.

    (b) 0 only.

    (c) 1 only.

    (d) -1 and 0.

    (e) -1 and 1

    (f) 0 and 1

    (g) -1, 0, and 1

    (h) f has no critical points.

    We calculate:
    $$f'(x) = \frac{-3(-e^{-x})}{(1 + e^{-x})^2} = \frac{3e^{-x}}{(1 + e^{-x})^2}.$$
    The top and bottom are both always positive, so there are no critical points, answer H.

11. Let $f(x) = \dfrac{3}{1 + e^{-x}}$. Find all inflection points of $f(x)$.

    (a) -1 only.

    (b) 0 only.

    (c) 1 only.

    (d) -1 and 0.

(e) -1 and 1

(f) 0 and 1

(g) -1, 0, and 1

(h) f has no inflection points.

From problem 10, we have

$$f'(x) = \frac{-3(-e^{-x})}{(1+e^{-x})^2} = \frac{3e^{-x}}{(1+e^{-x})^2}.$$

We continue calculating, using the quotient rule:

$$\begin{aligned} f''(x) &= \frac{3(-e^{-x})(1+e^{-x})^2 - 3e^{-x} \cdot 2(1+e^{-x})(-e^{-x})}{(1+e^{-x})^4} \\ &= \frac{3(-e^{-x})(1+e^{-x}) - 3e^{-x} \cdot 2(-e^{-x})}{(1+e^{-x})^3} \\ &= \frac{-3e^{-x}(1+e^{-x}) + 6e^{-x} \cdot e^{-x}}{(1+e^{-x})^3} \\ &= \frac{-3e^{-x} + 3e^{-2x}}{(1+e^{-x})^3} \\ &= \frac{3e^{-x}(-1+e^{-x})}{(1+e^{-x})^3}. \end{aligned}$$

Since e^{-x} and the bottom are both always positive, there is a possible inflection point only when $-1 + e^{-x} = 0$, hence, when $e^{-x} = 1$, i.e., when $x = 0$. Since f'' changes sign there, it is actually an inflection point, and we have answer B.

12. You have been asked to design a cylindrical tin cup. This cup will be shaped like a cylinder, with radius r and height h. Since it is a cup, it will have an open top (but closed bottom), for a total surface area of $2\pi rh + \pi r^2$.
Your boss, who has fond memories of calculus, wants each cup to hold 64π cubic cm. Determine the radius r that minimizes the surface area.

    (a) 0

    (b) 1

(c) 2

(d) 3

(e) 4

(f) 5

(g) 6

(h) No minimum exists.

We recall that the volume of a cylinder is $V = \pi r^2 h$, hence $64\pi = \pi r^2 h$, or $h = \frac{64}{r^2}$. We plug in to the given equation for surface area to get

$$A(r) = 2\pi r \cdot \frac{64}{r^2} + \pi r^2 = \frac{128\pi}{r} + \pi r^2.$$

We proceed to take the derivative an apply our optimization techniques:

$$A'(r) = -\frac{128\pi}{r^2} + 2\pi r = \frac{2\pi}{r^2}(-64 + r^3),$$

hence the only critical point is at $r = \sqrt[3]{64} = 4$.
Since A is defined for r on $(0, \infty)$, and since $A' > 0$ for $r > 4$ and $A' < 0$ for $r < 4$, we get that this is actually the unique minimum. Answer E.

Problems 13-17: True/false. Each problem is worth 2 points.

13. True/false: if f is a differentiable function, with $f(1) = 0$ and $f(3) = 4$, then there is some c between 1 and 3 with $f'(c) = 2$.

(a) True

(b) False

True, since 2 is the average rate of change, and by the MVT.

14. True/false: If f is a continuous function defined on $[3, 6]$, then f has an absolute maximum (on the interval $[3, 6]$).

(a) True

(b) False

True, by the EVT.

15. True/false: $y = x^3$ has a local minimum at $x = 0$.

    (a) True
    (b) False

    False. It has a critical point which is neither a max nor a min.

16. True/false: If $f'(x)$ is positive for x between 1 and 2, then $f(1) > f(2)$.

    (a) True
    (b) False

    False – actually, $f(1) < f(2)$.

17. True/false: If c is an inflection point, then it is a local maximum or minimum of f'.

    (a) True
    (b) False

    False is slightly better, since the definition of inflection points allows vertical asymptotes (which are not maxes or mins).
    Since we have not emphasized the precise definition of the words "inflection point", I gave full credit for either answer on this problem.

Part II (30 points): In each of the following problems, show your work clearly in the space provided. Partial credit will be given, and a correct answer without supporting work may not receive credit.

1. (7 points) Find the derivative of $x^{\sqrt{x}}$.

   We apply logarithmic differentiation techniques. Let $y = x^{\sqrt{x}}$. Thus

$$\begin{aligned} \ln y &= \ln x^{\sqrt{x}} = \sqrt{x} \ln x \\ \frac{y'}{y} &= \frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x} \end{aligned}$$

hence

$$y' = y \cdot (\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}) = x^{\sqrt{x}}(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}).$$

2. Let $f(x) = \dfrac{x^2}{x-2}$.

   (a) (1 point) Where does $f(x)$ fail to be continuous?
   At $x = 2$, when the bottom is 0.

   (b) (3 points) Calculate the first and second derivatives of f.

$$f'(x) = \frac{2x \cdot (x-2) - x^2 \cdot 1}{(x-2)^2} = \frac{x^2 - 4x}{(x-2)^2}$$

$$\begin{aligned} f''(x) &= \frac{(2x-4)\cdot(x-2)^2 - (x^2-4x)\cdot 2 \cdot (x-2) \cdot 1}{(x-2)^4} \\ &= \frac{(2x-4)\cdot(x-2) - (x^2-4x)\cdot 2}{(x-2)^3} \\ &= \frac{8}{(x-2)^3}. \end{aligned}$$

   (c) (3 points) Find all critical points and possible inflection points of f.
   Both f' and f'' are defined except at 2, where f is not defined either, so the only critical points and inflection points occur at 0s of f' and f''.
   The top of f' is $x(x-4)$, so the critical points are 0 and 4.
   The top of f'' is 8, which is never 0, so there are no inflection points.

   (d) (3 points) Describe the intervals where f is increasing/decreasing, and concave up/down.

	$(-\infty, 0)$	0	$(0,2)$	2	$(2,4)$	4	$(4,\infty)$
x	-	0	+	+	+	+	+
$(x-2)^2$	+	+	+	0	+	+	+
$(x-4)$	-	-	-	-	-	0	+
$f'(x)$	+	0	-	X	-	0	+

   So f is decreasing on $(0,2)$ and $(2,4)$, increasing everywhere else.

Similarly, for f'' we have

	$(-\infty, 2)$	2	$(2, \infty)$
$(x-2)^3$	-	0	+
$f'(x)$	-	X	+

(e) (2 points) Find all asymptotes of f.

By l'Hopital,

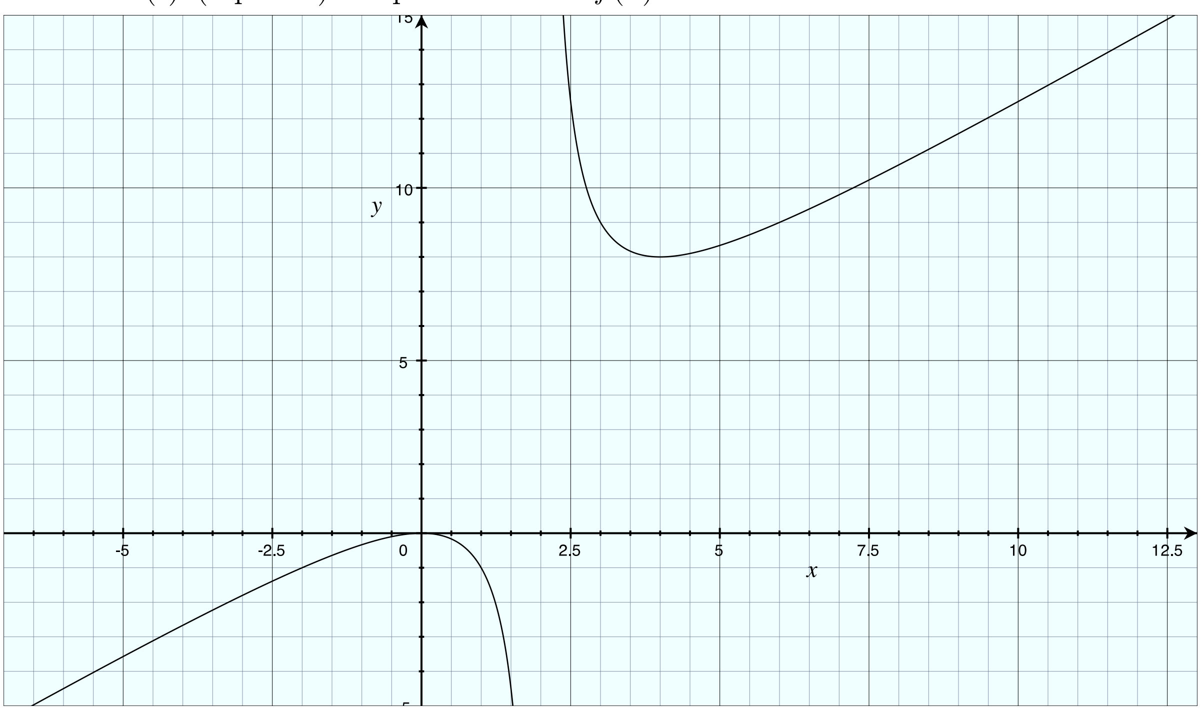
$$\lim_{x \to \pm\infty} \frac{x^2}{x-2} = \lim_{x \to \pm\infty} \frac{2x}{1} = \pm\infty,$$

so no horizontal asymptotes.

But

$$\lim_{x \to 2-} \frac{x^2}{x-2} = -\infty, \qquad \lim_{x \to 2+} \frac{x^2}{x-2} = +\infty$$

(f) (3 points) Graph a sketch of $f(x)$.



3. In our class, we calculated the derivative of e^x, and used inverse function techniques to find the derivative of $\ln x$. Suppose instead that you know that $\frac{d}{dx} \ln x = \frac{1}{x}$, but have not calculated $\frac{d}{dx} e^x$.

(a) (2 points) What is the inverse function of $y = e^x$?

As discussed in class, the inverse function for e^x is $\ln x$.

(a) (6 points) Using our inverse function differentiation techniques, and the derivative of $\ln x$, show that $\frac{d}{dx}e^x = e^x$. Let $y = e^x$.

Then

$$\begin{aligned} \ln y &= \ln e^x = x \\ \frac{y'}{y} &= 1 \qquad\qquad \text{hence} \\ y' &= y = e^x. \end{aligned}$$