Math 131
Final Examination Solutions – May 2, 2009

General Instructions: You may use a simple calculator that is not graphing or programmable. You may have up to four 3x5 cards, but no other notes.

Part I (60 points): For each of the following 17 problems, mark your answer on the answer card. For Part I, only the answer on the card will be graded.

Problems 1-10: Multiple choice. Each problem is worth 5 points.

1. Evaluate $\int_0^{\pi/6} \sin 3x \, dx$

   (a) 2
   (b) 3/2
   (c) 4/3
   (d) 1
   (e) 1/2
   (f) 1/3
   (g) 0
   (h) ∞

   We calculate:

   $$\int_0^{\pi/6} \sin 3x \, dx = \left[-\frac{1}{3}\cos 3x\right]_0^{\pi/6} = -\frac{1}{3}\cos\frac{\pi}{2} + \frac{1}{3}\cos 0 = \frac{1}{3}, \text{ answer F.}$$

2. Find the equation of the tangent line to $f(x) = \ln 2x + x^2 - \ln 2$ at the point $(1, 1)$.

   (a) $y = 1$
   (b) $y = 2.5(x - 2) + 1 = 2.5x - 4$
   (c) $y = 2.5(x - 1) + 2 = 2.5x - 0.5$

(d) $y = 2.5(x-1)+1 = 2.5x - 1.5$

(e) $y = 3(x-2)+1 = 3x-5$

(f) $y = 3(x-1)+2 = 3x-1$

(g) $y = 3(x-1)+1 = 3x-2$

(h) $x = 1$

We calculate $f'(x) = \frac{2}{2x} + 2x = \frac{1}{x} + 2x$. Then the tangent line is

$$y - 1 = f'(1)(x-1) = 3(x-1),$$

which is equivalent with answer G.

3. Find all points where the tangent line to the graph of $\dfrac{x^3}{x^2-3}$ is horizontal.

(a) 0, $\sqrt{3}$ and $-\sqrt{3}$

(b) 0 and $\sqrt{3}$

(c) 0, 3, and $\sqrt{3}$

(d) 0, 3, -3, $\sqrt{3}$, $-\sqrt{3}$

(e) 0, 3, and -3

(f) 0 and 3

(g) 0, π, $-\pi$

(h) No such points.

We start by calculating the derivative by the quotient rule

$$y' = \frac{3x^2 \cdot (x^2-3) - x^3 \cdot 2x}{(x^2-3)^2} = \frac{x^4 - 9x^2}{(x^2-3)^2}.$$

The top factors as $x^2(x-3)(x+3)$, and we see it has roots $0, 3, -3$. Since horizontal line has slope zero, we see these are the points with horizontal tangent, answer E.

4. Calculate $\lim_{x \to 0} \dfrac{x^2}{\sin(x^2 - x)}$

(a) 0
(b) 1
(c) -1
(d) 1/2
(e) π
(f) $-\infty$
(g) ∞
(h) undefined

We notice that the limit has the form $\frac{0}{0}$, so we apply l'Hopital.

$$\lim_{x\to 0} \frac{x^2}{\sin(x^2-x)} = \lim_{x\to 0} \frac{2x}{\cos(x^2-x)\cdot(2x-1)}.$$

We then try plugging in, which gives us $\frac{0}{1\cdot(-1)} = 0$. Answer A.

5. Calculate $\lim_{x\to 0} \frac{x^2}{e^x}$

(a) 0
(b) 1
(c) -1
(d) e^2
(e) e
(f) $-\infty$
(g) ∞
(h) undefined

The function is continuous at 0, so we plug in to get $\frac{0}{1} = 0$. Answer A.

6. Find the maximum value of $f(x) = e^x + e^{-x}$ on the interval $[-3, 3]$.

(a) $e^3 - e^{-3}$

(b) $2\ln 3$

(c) 2

(d) 21

(e) $e + e^{-1}$

(f) 30

(g) $e^3 + e^{-3}$

(h) ∞

We calculate the derivative

$$f'(x) = e^x - e^{-x}.$$

We pull out the e^{-x} to write

$$f'(x) = e^{-x} \cdot (e^{2x} - 1).$$

Since e^{-x} is strictly positive, the only possible root is when $e^{2x} = 1$, i.e., when $x = 0$. We evaluate the function at the critical point and endpoints. At the critical point the value is 2, but at the endpoints it is $e^3 + e^{-3}$. Since $e^3 + e^{-3} > 2$, we see that the answer is G.

7. Which of the following is the Riemann sum with a uniform partition and right endpoints representing $\int_1^4 \ln x \, dx$?

(a) $\displaystyle\sum_{k=1}^{n} \ln(1 + \frac{k}{n}) \cdot \frac{1}{n}$

(b) $\displaystyle\sum_{k=1}^{n} \ln(1 + \frac{4k}{n}) \cdot \frac{4}{n}$

(c) $\displaystyle\sum_{k=1}^{n} \ln(\frac{4k}{n}) \cdot \frac{4}{n}$

(d) $\displaystyle\sum_{k=1}^{n} \ln(1 + \frac{k-1}{n}) \cdot \frac{1}{n}$

(e) $\displaystyle\sum_{k=1}^{n} \ln(\frac{k}{n}) \cdot \frac{1}{n}$

(f) $\sum_{k=1}^{n} \ln(\frac{4(k-1)}{n}) \cdot \frac{4}{n}$

(g) $\sum_{k=1}^{n} \ln(1 + \frac{3k}{n}) \cdot \frac{3}{n}$

(h) $\ln 4 - \ln 1$

We recognize the answer as G, by setting up the Riemann sum. Alternately, you might recognize that the length of $[1, 4]$ is 3, which lets you eliminate all of the other possibilities.

8. Which of the following is the ϵ-δ definition of the statement

$$\lim_{x \to 0} e^x = 1?$$

(a) for all $\epsilon > 0$ there exists a $\delta > 0$ such that $0 < |x - 1| < \delta \implies |e^x| < \epsilon$.

(b) for all $\epsilon > 0$ there exists a $\delta > 0$ such that $0 < x - 1 < \delta \implies |e^x| < \epsilon$.

(c) for all $\epsilon > 0$ there exists a $\delta > 0$ such that $0 < |x| < \delta \implies |e^x - 1| < \epsilon$.

(d) for all $\epsilon > 0$ there exists a $\delta > 0$ such that $0 < x < \delta \implies |e^x - 1| < \epsilon$.

(e) for all $\epsilon > 0$ there exists a $\delta > 0$ such that blah blah blah math greek blah blah.

(f) for all $\epsilon < 0$ there exists a $\delta < 0$ such that $0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$.

(g) Undefined/doesn't exist.

(h) Rabbit.

By plugging into the definition of limit, we get answer C.

9. Let $f(x) = e^{x^2}$, and $F(x)$ be any antiderivative of f. Find all critical points of F.

(a) e, 0, and e^{-1} only.
(b) 0 and e^{-1} only.
(c) e and 0 only
(d) e and e^{-1} only.
(e) e^{-1} only.
(f) e only.
(g) 0 only
(h) F has no critical points.

Since $F'(x) = f(x)$, critical points of F are zeros of f, or points where f is undefined. But e^{x^2} is always defined and positive, hence there are no critical points. Answer H.

10. At what points does $f(x) = \dfrac{(x-2)\sin x}{x^2 - x}$ have a vertical asymptote?

(a) At 1, 0, and -1 only.
(b) At 0 and -1 only.
(c) At 1 and 0 only
(d) At 1 and -1 only.
(e) At -1 only.
(f) At 1 only.
(g) At 0 only
(h) $f(x)$ has no vertical asymptotes.

Since the bottom is 0 at $x = 0$ and 1, the function is discontinuous at 0 and 1. We see that

$$\lim_{x\to 1+} \frac{(x-2)\sin x}{x^2 - x} = -\infty, \text{ since } \frac{(-1)\cdot \sin 1}{\text{small positive}} = \text{large negative},$$

hence 1 is a vertical asymptote. On the other hand, we recall that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, hence

$$\lim_{x\to 0} \frac{(x-2)\sin x}{x^2 - x} = \lim_{x\to 0} \frac{(x-2)}{(x-1)} \cdot \frac{\sin x}{x} = \frac{(-2)}{(-1)} \cdot 1 = 2.$$

Thus, 0 is <u>not</u> a vertical asymptote. Answer F.

Problems 11-15: True/false. Each problem is worth 2 points.

11. True/false: if f is an increasing function on $(-\infty, 1)$ and a decreasing function on $(1, \infty)$, then the derivative of f is defined at 1 and $f'(1) = 0$.

    (a) True

    (b) False

    False. An example of such a function with the derivative undefined is $-|x-1|$.

12. True/false: If f is continuous at 3, then the limit $\lim_{x \to 3+} f(x)$ exists.

    (a) True

    (b) False

    True. If f is continuous at 3, then $\lim_{x \to 3+} f(x) = \lim_{x \to 3} f(x) = f(3)$.

13. True/false: a Riemann sum $\sum_{k=1}^{n} f(c_k)\Delta x_k$ for f on $[a, b]$ is an antiderivative for f on the same interval.

    (a) True

    (b) False

    False. A Riemann sum evaluates to a number, which is never an antiderivative. There is a relationship between antiderivatives and definite integrals (the FTC), but it is more complicated than this!!

14. True/false: The definite integral $\int_0^3 (2 - e^x)\, dx$ represents an area.

    (a) True

    (b) False

False. $(2 - e^x)$ is positive at 0, negative at 3 (since $2 - e^3 < 0$), so the definite integral does not represent an area.

15. True/false: if f is differentiable everywhere, then it has an antiderivative.

    (a) True

    (b) False

    True. If f is differentiable everywhere, then it is continuous, and the FTC tells us that continuous functions have an antiderivative.

Part II (40 points): In each of the following problems, show your work clearly in the space provided. Partial credit will be given, and a correct answer without supporting work may not receive credit.

1. (3 points) In 2-4 sentences, explain why limits are an important concept in Math 131.

   The limit is fundamental for Math 131, since the derivative and definite integral are defined as certain limits:

   $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

   and

   $$\lim_{|P| \to 0} \sum_{k=1}^{n} f(c_k) \Delta x_k.$$

   (Grading scheme: Full credit required at least one half of this connection. I gave 2 points for convincing discussion of asymptotes, hole-filling, etc; and 0-1 points for less convincing discussion.)

2. Consider the function $y(t) = e^t + e^{-t}$.

   (a) (4 points) Where is $y(t)$ rising? Where is it falling?

   We calculate $y'(t) = e^t - e^{-t}$. We write as $y'(t) = e^{-t}(e^{2t} - 1)$, and since e^{-t} is always positive, the only critical point is when $e^{2t} = 1$,

i.e., at $t = 0$.
By plugging in 1 and -1, we see that $y' < 0$ on $(-\infty, 0)$, so y is decreasing there. Conversely, $y' > 0$ on $(0, \infty)$, so y is increasing on this interval.
Compare with problem 6.

(b) (4 points) Where is $y(t)$ concave upward? Concave downward?

We calculate $y''(t) = e^t + e^{-t}$. Since both e^t and e^{-t} are always positive, so is there sum. Thus, y'' is always positive, and y is always concave up.

(c) (3 points) What is $\lim_{t\to\infty} y(t)$? What is $\lim_{t\to-\infty} y(t)$?

$$\begin{aligned}
\lim_{t\to\infty} e^t + e^{-t} &= \infty + 0 = \infty \\
\lim_{t\to-\infty} e^t + e^{-t} &= 0 + \infty = \infty
\end{aligned}$$

(d) (3 points) Graph the above function $y(t)$. Identify on your graph all critical points and inflection points, as well as the limits from part (c).

(graph omitted)

3. A circus performer throws a knife into the air. Gravity provides a constant acceleration of $-4.9\ \mathrm{m/s^2}$. The initial velocity at $t = 0$ is 6 m / s, and height at $t = 0$ is 1 m.

(a) (6 points) Find the vertical velocity $v(t)$ and height $s(t)$ of the knife at time t.

We are given $a(t) = -4.9$. Since $v(t)$ is an antiderivative of $a(t)$, we get that $v(t) = -4.9t + C$. Since $v(0) = -4.9 \cdot 0 + C = 6$, we see that $C = 6$, hence

$$v(t) = -4.9t + 6.$$

We repeat this process to find $s(t)$: since it is an antiderivative of $v(t)$, we have

$$s(t) = -\frac{4.9}{2}t^2 + 6t + C,$$

and as $s(0) = 1$, we get

$$s(t) = -\frac{4.9}{2}t^2 + 6t + 1.$$

Note: The usual constant of gravity is -9.8, but as everyone knows, gravity works differently in the circus!

(b) (3 points) Find all critical points of $s(t)$. When is the knife at its highest point?

The critical point is when $v(t) = -4.9t + 6 = 0$, hence at $t = \frac{6}{4.9}$. Since $a = -4.9$ is negative, this is a max.

(c) (2 points) At what time does the knife hit the ground? (height 0)

We solve

$$s(t) = -\frac{4.9}{2}t^2 + 6t + 1 = 0$$

using the quadratic equation, and get

$$t = \frac{-6 \pm \sqrt{6^2 - 4 \cdot (-\frac{4.9}{2}) \cdot 1}}{-4.9} \approx 2.61, -0.16.$$

Since the knife is thrown at $t = 0$, we exclude the negative value as outside our domain, and see that the knife hits after about 2.61 seconds.

4. (12 points) Calculate the following, showing all your work:

(a) $\int \sin x \cdot \cos^2 x \, dx$

We perform substitution with $u = \cos x$, so that $du = -\sin x \, dx$, so

$$\int \sin x \cdot \cos^2 x \, dx = \int u^2 \, (-1) du = -\frac{u^3}{3} + C = -\frac{\cos^3 x}{3} + C.$$

(b) $\displaystyle\int \frac{e^{2x}}{1-e^x}\,dx$

We perform substitution with $u = 1 - e^x$, so that $du = e^x\,dx$. Then $dx = \frac{1}{e^x}\,du$, so we substitute and get

$$\int \frac{e^{2x}}{1-e^x}\,dx = \int \frac{e^{2x}}{u}\frac{1}{e^x}du = \int \frac{e^x}{u}\,du.$$

This leaves us with an extra e^x, which we need to write in terms of u to integrate. Fortunately, $u = 1 - e^x$, so $e^x = 1 - u$, so the integral becomes

$$\int \frac{1-u}{u}\,du = \int \frac{1}{u} - 1\,du = \ln|u| - u + C = \ln|1 - e^x| - e^x + C.$$