Math 131

Exponentials/Logarithms Overview – February 18, 2009

1. Definition
   (a) e is a real number (about 2.7).
   (b) $e^3 = e \cdot e \cdot e$, and $e^{0.5} = \sqrt{e}$. So $e^{3/2} = \sqrt{e^3}$.
   (c) e^x can be similarly defined for all rational numbers, and filled in for irrational numbers by taking limits.
   (d) $\ln x$ is the inverse function of e^x. This means that $\ln e^x = e^{\ln x} = x$.

2. Useful values and limits.

$$\begin{aligned} e^0 &= 1, \qquad \lim_{x\to\infty} e^x = \infty, \qquad \lim_{x\to-\infty} e^x = 0 \\ \ln 1 &= 0, \qquad \ln e = 1, \qquad \lim_{x\to 0+} \ln x = -\infty, \qquad \lim_{x\to\infty} \ln x = \infty. \end{aligned}$$

   Note: $\ln x$ is not defined for $x \leq 0$!

3. Change of base
   (a) $2^x = (e^{\ln 2})^x = e^{x\cdot\ln 2}$
   (b) $\log_2 x = \dfrac{\ln x}{\ln 2}$

   In both formulas, 2 can be replaced by any positive number.

4. Derivatives and integrals

$$\begin{aligned} \frac{d}{dx}e^x &= e^x, \qquad \frac{d}{dx}\ln x = \frac{1}{x} \\ \int e^x\,dx &= e^x + C \qquad \int \frac{1}{x}\,dx = \ln|x| + C \end{aligned}$$

5. Derivatives in other bases:

   We consider e^x and $\ln x$ rather than, say, 2^x and $\log_2 x$ because their derivatives and integrals have these natural forms. Compare with the derivatives of 2^x and $\log_2 x$:

$$\begin{aligned} \frac{d}{dx}2^x &= \frac{d}{dx}e^{x\cdot\ln 2} = \ln 2 \cdot e^{x\cdot \ln 2} = \ln 2 \cdot 2^x \\ \frac{d}{dx}\log_2 x &= \frac{d}{dx}\frac{\ln x}{\ln 2} = \frac{1}{x\cdot \ln 2} \end{aligned}$$

6. Fundamental identities:

$$\begin{aligned} e^{x+y} &= e^x e^y \qquad \text{(i.e., } e^x \text{ "turns + into } \cdot \text{ ")} \\ e^{xy} &= (e^x)^y \\ \ln xy &= \ln x + \ln y \qquad \text{(i.e., } \ln x \text{ "turns } \cdot \text{into + ")} \\ \ln x^y &= y \ln x \end{aligned}$$

7. Graphs.
   (The graph of $\ln x$ is that of e^x flipped over the line $y = x$!)