BRIEF NOTES ON DIFFERENTIATING UNDER THE INTEGRAL

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Recall that the regularity condition (R4) from Hogg-McKean-Craig says

$$\frac{\partial}{\partial\theta}\left[\int_a^b f(x,\theta)\,dx\right] = \int_a^b \frac{\partial}{\partial\theta}\left[f(x,\theta)\right]\,dx \qquad \text{for all } a,b \in [-\infty,\infty],$$

and the same for the 2nd derivative.

(We are abusing notation writing a closed interval with ∞, but the meaning is clear, and can be made to have precise meaning via Math 417.)

1. Antiexample

For $\theta \in (0, \frac{1}{4})$, let

$$f(x,\theta) = \begin{cases} x & \text{for } 0 \le x \le \sqrt{\theta} \\ -x + 2\sqrt{\theta} & \text{for } \sqrt{\theta} \le x \le 2\sqrt{\theta} \\ 0 & \text{else.} \end{cases}$$

Notice that $f(x,\theta)$ is continuous, and differentiable in θ for any fixed x. Then

$$\begin{aligned} \frac{\partial}{\partial\theta}\left[\int_0^1 f(x,\theta)\,dx\right] &= \frac{\partial}{\partial\theta}\left[\int_0^{\sqrt{\theta}} x\,dx + \int_{\sqrt{\theta}}^{2\sqrt{\theta}} -x + 2\sqrt{\theta}\,dx\right] \\ &= \frac{\partial}{\partial\theta}\left[\frac{\theta}{2} + (-\frac{4\theta}{2} + 4\theta) - (-\frac{\theta}{2} + 2\theta)\right] \\ &= \frac{\partial}{\partial\theta}\theta = 1, \end{aligned}$$

but $\frac{\partial}{\partial\theta}f(x,\theta)$ is $\frac{1}{\sqrt{\theta}}$ for $\sqrt{\theta} \le x \le 2\sqrt{\theta}$ (and is 0 otherwise), so

$$\int_0^1 \frac{\partial}{\partial\theta}\left[f(x,\theta)\right]\,dx = \int_{-\sqrt{\theta}}^{\sqrt{\theta}} \frac{1}{\sqrt{\theta}}\,dx = (\frac{\sqrt{\theta}}{\sqrt{\theta}} + \frac{\sqrt{\theta}}{\sqrt{\theta}}) = 1.$$

Since $4 \neq 1$, the given function fails (R4).

As shown in class, if you rescale $f(x,\theta)$ to be a pdf with respect to x for each θ, it still fails (R4).

2. Sufficient conditions

Theorem 1. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous as a function of x and θ, and $\frac{\partial}{\partial\theta}f(x,\theta)$ is also continuous as a function of x and θ, then*

$$\frac{\partial}{\partial\theta}\left[\int_a^b f(x,\theta)\,dx\right] = \int_a^b \frac{\partial}{\partial\theta}\left[f(x,\theta)\right]\,dx$$

for all $a,b \in [-\infty,\infty]$.

3. Application and cultural note

When the conditions for differentiating under the integral sign are met, it can be a powerful tool for evaluating integrals. For example,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-tx^2/2}\, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2(1/\sqrt{t})^2}}\, dx = \frac{1}{\sqrt{t}},$$

hence

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial t} e^{-tx^2/2} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot (-\frac{x^2}{2}) \cdot e^{-tx^2/2}\, dx,$$

which is equal to

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-tx^2/2}\, dx = \frac{\partial}{\partial t} \frac{1}{\sqrt{t}} = -\frac{1}{2t^{3/2}}$$

Plugging in $t = 1$, we observe the pdf of the standard normal random variable Z in the original integral, and the integral for $E(-\frac{1}{2}Z^2)$ in the derivative. Thus, we have found a new proof that $E(Z^2) = 1$.

Higher even moments can be calculated by taking additional derivatives.

The late great Richard Feynman favored this technique, as he discusses in his memoir, *Surely You're Joking Mr. Feynman*[1].

> One day he told me to stay after class. "Feynman," he said, "you talk too much and you make too much noise. I know why. You're bored. So I'm going to give you a book. You go up there in the back, in the corner, and study this book, and when you know everything that's in this book, you can talk again." So every physics class, I paid no attention to what was going on with Pascal's Law, or whatever they were doing. I was up in the back with this book: Advanced Calculus, by Woods. Bader knew I had studied Calculus for the Practical Man a little bit, so he gave me the real works—it was for a junior or senior course in college. It had Fourier series, Bessel functions, determinants, elliptic functions—all kinds of wonderful stuff that I didn't know anything about. That book also showed how to differentiate parameters under the integral sign—it's a certain operation. It turns out that's not taught very much in the universities; they don't emphasize it. But I caught on how to use that method, and I used that one damn tool again and again. So because I was self-taught using that book, I had peculiar methods of doing integrals. The result was, when guys at MIT or Princeton had trouble doing a certain integral, it was because they couldn't do it with the standard methods they had learned in school. If it was contour integration, they would have found it; if it was a simple series expansion, they would have found it. Then I come along and try differentiating under the integral sign, and often it worked. So I got a great reputation for doing integrals, only because my box of tools was different from everybody else's, and they had tried all their tools on it before giving the problem to me.

As a major in a technical field, you should certainly read this highly entertaining book in its entirety.

References

[1] Richard Feynman, *Surely You're Joking, Mr. Feynman!: Adventures of a Curious Character*, W. W. Norton & Co, 1988.