1. Working with inequalities

Three useful rules for working with inequalities are:

(1) $|a \cdot b| = |a| \cdot |b|$
(2) $|a + b| \leq |a| + |b|$ (the triangle inequality)
(3) $|a - b| \leq |a| + |b|$

Rule (3) may seem surprising. Let us see how to prove it. We calculate:

$$|a - b| = |a + (-b)| \leq |a| + |-b| = |a| + |b|.$$ 

2. Bounds

Definition 1.
An upper bound for a function $f$ is a number $U$ so that: for all $x$, we have $f(x) \leq U$.
A lower bound for a function $f$ is a number $L$ so that: for all $x$, we have that $f(x) \geq L$.

A bound in absolute value, which is what we will usually refer to as just a bound, is a number $M$ so that $|f(x)| \leq M$ for all $x$.

Notice that if $M$ is a bound in absolute value for $f$, then $-M$ and $M$ are lower and upper bounds for $f$, and conversely that if $L$ and $U$ are lower and upper bounds, then $\max(|L|, |U|)$ is a bound for $f$ in absolute value.

We’ll usually be interested in bounds in absolute value, since they are convenient and quick to work with.

Definition 2. We say $f$ has an upper bound $U$ on the interval $[a, b]$ if: for all $x$ on $[a, b]$, we have $f(x) \leq U$. Similarly for lower bounds and bounds in absolute value.

Example 3. Some bounds:

(1) $|\sin x| \leq 1$ for all $x$. Thus, 1 is a bound (in absolute value) for $\sin x$. So is 2, and 3, and 3.17, but not 0.98.
(2) $|x^3| \leq 27$ on the interval $[-3, 1]$.
(3) $|e^x| \leq e^2$ on the interval $[-5, 2]$. We could also say that $|e^x| \leq 9$ on the interval $[-5, 2]$, since $9 > e^2$.

2.1. Geometric interpretation. Geometrically, an upper bound is a horizontal line that the graph of the function does not go above. Similarly, a lower bound is a horizontal line that the graph does not go below. A bound in absolute value ‘traps’ the graph of the function in a band between the horizontal lines $y = -M$ and $y = M$.

3. Example

We find a bound for $12\sin x^2 - x \cos x$ on the interval $[-3, 2]$. Using the rules from Part 1, we break it apart:

$$|12\sin x^2 - x \cos x| \leq |12\sin x^2| + |x \cos x|$$
$$= 12|\sin x^2| + |x| \cdot |\cos x|.$$
Since \( |x| \leq 3 \) on the interval \([-3, 2]\), and since \( |\sin*| \leq 1 \) and \( |\cos*| \leq 1 \), we get that
\[
|12\sin x^2 - x\cos x| \leq 12 \cdot 1 + 3 \cdot 1 = 15
\]
(on the interval \([-3, 2]\)).

**Note:** this is not the least possible bound. However, for many applications it is good enough.

### 4. ERROR BOUNDS FOR NUMERICAL INTEGRATION

To find an error bound for the Trapezoid Rule or Simpson’s rule, we first need to find an upper bound for \( |f''(x)| \) or \( |f^{(4)}(x)| \) (respectively) on the interval in question.

**Example 4.** Find a number of subintervals so that the error of the Trapezoid Rule applied to \( \int_0^1 \sin x^2 \, dx \) is at most 0.01.

**Solution:** We calculate derivatives:
\[
f(x) = \sin(x^2), \quad f'(x) = \cos(x^2) \cdot 2x \quad \text{and} \quad f''(x) = -\sin(x^2) \cdot 4x^2 + \cos(x^2) \cdot 2 = -4x^2 \sin(x^2) + 2 \cos(x^2).
\]
So
\[
|f''(x)| \leq |4x^2 \sin(x^2)| + |2 \cos(x^2)|
= 4 \cdot |x^2| \cdot |\sin(x^2)| + 2 \cdot |\cos(x^2)|.
\]
As previously discussed, \( |\sin(\text{anything})| \leq 1 \), and similarly for cos. Since we are integrating on the interval \([0, 1]\), we have \( 0 \leq x \leq 1 \) and in particular \( |x^2| \leq 1 \). Thus,
\[
|f''(x)| \leq 4 \cdot 1 \cdot 1 + 2 \cdot 1 = 6.
\]
Plugging into the error formula for the Trapezoid Rule, we get that
\[
|E_T| \leq \frac{M \cdot (b - a)^3}{12n^2} = \frac{6 \cdot (1 - 0)^3}{12n^2} = \frac{1}{2n^2}.
\]
So to have \( |E_T| \leq 0.1 \), it suffices to set \( \frac{1}{2n^2} \leq 0.1 \). Solving this, we get \( 2n^2 \geq 10 \), i.e., \( n \geq \sqrt{5} \).

Since \( 3^2 > 5 \), \( n = 3 \) will give already give the desired accuracy!

**Exercise 5.** How many subintervals do you want in order to force the error of the Trapezoid Rule to be less than 0.01 for \( \int_0^1 \sin x^2 \, dx \)?

**Exercise 6.** How many subintervals do you want in order to force the error of the Trapezoid Rule to be less than 0.01 for each of the following integrals?
\[
\int_{-2}^3 e^{x^2} \, dx, \quad \int_0^\sqrt{\pi} \cos(x^2) \, dx
\]

**Exercise 7.** For the integrals from Exercise 6, how many subintervals to force the error of Simpson’s Rule to be less than 0.01?