Math 132
Midterm Examination 2 Solutions – March 26, 2012

6 multiple choice, 4 long answer. 100 points.

Part I was multiple choice. Only the correct answers are listed here.

1. Find the Trapezoid Rule approximation using 4 subintervals of

$$\int_{-1}^{1} x^2\,dx.$$

   (f) 3/4

2. Find the Simpson's Rule approximation using 4 subintervals of

$$\int_{-1}^{1} x^2\,dx.$$

   (e) 2/3

3. Consider the system consisting of 3 point masses:
   10 kg at $(3,-1)$
   20 kg at $(2,10)$
   100 kg at $(1,0)$
   The center of mass is:

   (g) $(\frac{17}{13},\frac{19}{13})$

4. Simpson's Rule applied to the integral $\displaystyle\int_1^e \frac{1}{x}\,dx$ with $n = 20$ will be closest to:

   (k) 1
   (Since $\frac{1}{x} \leq 1$ on $[1,e]$, and then the error bound of $\frac{1\cdot(e-1)^5}{20^4}$ is quite small.)

5. Find the average value of $\sin x$ over the interval $[0,\pi]$.

   (d) $2/\pi$

6. The decay of a certain radioactive isotope of the element rabbitonium is governed by the differential equation $y' = -ky$. At $t = 0$ you have 300 mg of radioactive rabbitonium. At $t = 45$ minutes, you are left with only 100 mg of radioactive rabbitonium.
Then k is ______ per minute.

   (f) $\ln 3/45$.

Part II was long answer.

1. Differential equations

   (a) (8 points) Solve the differential equation $y' = x + xy$ subject to the initial condition $y(0) = 5$.

   Separating the equation, we have

$$\frac{y'}{1+y} = x$$

   hence

$$\begin{aligned} \int \frac{1}{1+y}\,dy &= \int x\,dx \\ \ln||1+y| &= \frac{x^2}{2} + C \\ 1+y &= Ae^{x^2/2} \\ y &= Ae^{x^2/2} - 1. \end{aligned}$$

   The initial condition $y(0) = 5 = Ae^0 - 1$ gives that $A = 6$, so

$$y = 6e^{x^2/2} - 1.$$

   (b) (8 points) At time $t = 0$, there is 1000 liters of water in a tank, with 80 kg of salt dissolved in it. Distilled water flows into the tank at 10 L/min, and water flows out of the tank at the same rate. The tank is continually stirred, and the salt is kept mixed evenly through the tank.
   Set up a differential equation (you needn't solve it) for the mass of salt in the tank at time t. (Your answer should be of the form $y' =$ ____.)

   Inflow of salt = 0,
   outflow of salt = (amount of salt in tank/1000) $\cdot$ 10,
   so if y = amount of salt in tank, then

$$y' = -\frac{y \cdot 10}{1000}.$$

   The initial condition is $y(0) = 80$.

2. Arc lengths and approximate integration

(a) (6 points) Set up a definite integral representing the length of the curve $y = x^3$ between $x = 0$ and $x = 4$.

$$\int_0^4 \sqrt{1 + (3x^2)^2}\, dx.$$

(b) (10 points) The first several derivatives of $f(x) = \sqrt{1+x^2}$ are as follows:

$$f'(x) = \frac{x}{\sqrt{1+x^2}}, \qquad f''(x) = \frac{1}{(1+x^2)^{3/2}}, \qquad f^{(3)}(x) = \frac{-3x}{(1+x^2)^{5/2}},$$
$$f^{(4)}(x) = \frac{12x^2-3}{(1+x^2)^{7/2}}, \qquad f^{(5)}(x) = \frac{45x - 60x^3}{(x^2+1)^{9/2}}.$$

Find (with justification) an n such that the Simpson's Rule approximation S_n for $\int_{-1}^{4} \sqrt{1+x^2}\, dx$ has error at most 0.001.

The main step in this problem is finding an upper bound for $f^{(4)}$.
Approach 1 to bounding $f^{(4)}$: (triangle inequality)
We have that

$$|f^{(4)}(x)| = \frac{|12x^2 - 3|}{|1+x^2|^{7/2}} \le \frac{12\,|x^2| + 3}{|1+x^2|^{7/2}}.$$

The top is $\le 12 \cdot 4^2 + 3$ on $[-1, 4]$, and the bottom is ≥ 1 everywhere, hence $\left|f^{(4)}(x)\right| \le \frac{12 \cdot 16 + 3}{1} = 195$.
Approach 2 to bounding $f^{(4)}$: (take another derivative)
The 5th derivative is continuous on $[1, 4]$, and has roots at 0 and $\pm\frac{\sqrt{3}}{2}$. We approximate these points and the endpoints, using the triangle inequality to simplify:

$$\begin{aligned}
\left|f^{(4)}(-1)\right| &= \frac{12-3}{(1+1)^{7/2}} \le \frac{9}{2^{6/2}} = \frac{9}{8} \\
\left|f^{(4)}(-\frac{\sqrt{3}}{2})\right| &= \frac{12 \cdot \frac{3}{4} - 3}{(1+\frac{3}{4})^{7/2}} = \frac{6}{(\frac{7}{4})^{7/2}} \le \frac{6}{(\frac{3}{2})^{6/2}} = \frac{16}{9} \\
\left|f^{(4)}(0)\right| &= \frac{3}{1^{7/2}} = 3 \\
\left|f^{(4)}(\frac{\sqrt{3}}{2})\right| & \quad \text{is the same as } \left|f^{(4)}(-\frac{\sqrt{3}}{2})\right| \\
\left|f^{(4)}(4)|\right| &= \frac{12 \cdot 16 - 3}{(1+16)^{7/2}} \le \frac{189}{16^{7/2}} = \frac{189}{4^7} \le 1
\end{aligned}$$

Since the max of $|f^{(4)}(x)|$ on $[-1, 4]$ occurs at one of the above points (as it is clearly zero at the points where it fails to be differentiable), we have that

$$\left|f^{(4)}(x)\right| \le 3$$

Finding the bound: Using the error bound for Simpson's rule, and letting M be as found in Approach 1 or Approach 2, we want

$$\frac{M \cdot (4-(-1))^5}{180n^4} \leq \frac{1}{10^3},$$

so that

$$n \geq \sqrt[4]{\frac{10^3 \cdot M \cdot 5^5}{180}}.$$

Writing that you take n to be the least even integer greater than this value (plugging in M to be 195, or 3, or whatever bound you found) gets full credit.

Finding an integer value for n (optional): We can factor and round up to find an n that "works". We showed that it suffices to take

$$n \geq \sqrt[4]{\frac{10^3 \cdot M \cdot 5^5}{180}} = \sqrt[4]{\frac{2^3 \cdot 5^8 \cdot M}{2^2 \cdot 3^2 \cdot 5}} = \sqrt[4]{\frac{2 \cdot 5^7 \cdot M}{3^2}}.$$

If we followed approach 1, then it is convenient to notice that $195 \leq 200$ (as 200 has a very nice factorization).

$$\sqrt[4]{\frac{2 \cdot 5^7 \cdot M}{3^2}} = \sqrt[4]{\frac{2 \cdot 5^7 \cdot 195}{3^2}} \leq \sqrt[4]{\frac{2 \cdot 5^7 \cdot 200}{3^2}} = \sqrt[4]{\frac{2^4 \cdot 5^{10}}{3^2}} = 2 \cdot 5^2 \cdot \frac{\sqrt{5}}{9} \leq 50,$$

and we see that $n = 50$ suffices. (Similarly for approach 2.)

3. Calculations

(a) (6 points) Find an upper bound for $\left|2e^{-(x+1)^2} + 12\sin(x+1)^2\right|$ on the interval $[-3, 3]$.

Using the triangle inequality,

$$\begin{aligned} \left|2e^{-(x+1)^2} + 12\sin(x+1)^2\right| &\leq \left|2e^{-(x+1)^2}\right| + \left|12\sin(x+1)^2\right| \\ &= 2|e^{-(x+1)^2}| + 12|\sin(x+1)^2| \\ &\leq 2 \cdot 1 + 12 \cdot 1 = 14. \end{aligned}$$

(b) (7 points) Evaluate $\int x^2 \cos x \, dx$.

We apply integration by parts 2 times. First, take $u_1 = x^2$ and $dv_1 = \cos x \, dx$, so that $du_1 = 2x \, dx$ and $v_1 = \sin x$. We get

$$\int x^2 \cdot \cos x \, dx = x^2 \cdot \sin x - \int 2x \cdot \sin x \, dx.$$

Then take $u_2 = 2x$ and $dv_2 = \sin x \, dx$, so that $du_2 = 2 \, dx$ and $v_2 = -\cos x$. We get the integral to be

$$= x^2 \sin x + 2x \cos x - \int 2\cos x \, dx = x^2 \sin x + 2x \cos x - 2\sin x + C.$$

(c) (6 points) Evaluate $\displaystyle\int_0^1 \frac{x}{1+x^2}\,dx$.

We substitute $u = 1 + x^2$, so that $du = 2x\,dx$, and the integral becomes

$$\int_0^1 \frac{x}{1+x^2}\,dx = \int_1^2 \frac{1}{u}\frac{du}{2} = \left[\frac{\ln|u|}{2}\right]_1^2 = \frac{\ln 2}{2} - 0.$$

(d) (6 points) Evaluate $\displaystyle\int_{-1}^1 x\tan^{-1} x\,dx$.

We apply integration by parts with $u = \tan^{-1} x$ and $dv = x\,dx$, so that $du = \frac{1}{1+x^2}\,dx$ and $v = \frac{x^2}{2}$. We get

$$\int_{-1}^1 x\tan^{-1} x\,dx = \left[\tan^{-1} x \cdot \frac{x^2}{2}\right]_{-1}^1 - \int_{-1}^1 \frac{x^2}{2} \cdot \frac{1}{1+x^2}\,dx.$$

We notice that $\frac{1}{2}\frac{x^2}{1+x^2} = \frac{1}{2}\left(1 - \frac{1}{1+x^2}\right)$, hence the integral is

$$= \left[\tan^{-1} x \cdot \frac{x^2}{2}\right]_{-1}^1 + \frac{1}{2}\int_{-1}^1 \frac{1}{1+x^2} - 1\,dx = \left[\tan^{-1} x \cdot \frac{x^2}{2} + \frac{1}{2}\tan^{-1} x - \frac{x}{2}\right]_{-1}^1$$

$$= \left(\frac{\pi}{4}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\right) - \left(\left(-\frac{\pi}{4}\right)\cdot\frac{1}{2} - \frac{1}{2}\cdot\frac{\pi}{4} + \frac{1}{2}\right) = \frac{\pi}{2} - 1.$$

4. Volumes and centroids

In both problems on this page, we consider the region between the x-axis and the graph of $y = e^x$ for $0 \le x \le 2$.

(a) (11 points) Find the volume of the solid formed by rotating the given region around the y-axis.

Solution 1: (easier) We use cylindrical shells:

$$V = 2\pi \cdot \int_0^2 x \cdot e^x\,dx = 2\pi\left[xe^x\right]_0^2 - 2\pi\int_0^2 e^x = 2\pi\left[xe^x - e^x\right]_0^2 = 2\pi(e^2+1).$$

Solution 2: (harder, sketched only) We use discs. The shape is between $x = \ln y$ and $x = 2$ for $1 \le y \le e^2$, and between $x = 0$ and $x = 2$ for $0 \le y \le 1$. Thus, we get

$$V = \pi\int_0^1 2^2\,dy + \pi\int_1^{e^2} (\ln y)^2\,dy.$$

The integral of $(\ln y)^2$ may be computed by two applications of integration by parts.

(b) (8 points) Find the center of mass $\overline{x}$ with respect to x of the solid formed by rotating the given region around the <u>x-axis</u>.
<u>Half credit</u> will be received for instead finding the center of mass $\overline{x}$ of the given (unrotated) region.

Full credit: Assume uniform density 1. The density with respect to x is the cross-sectional area $A(x) = \pi(e^x)^2 = \pi e^{2x}$, hence we have

$$\overline{x} = \frac{\int_0^2 x \cdot A(x)\, dx}{\int_0^2 A(x)\, dx} = \frac{\int_0^2 x \cdot \pi e^{2x}\, dx}{\int_0^2 \pi e^{2x}\, dx} = \frac{\int_0^2 x \cdot e^{2x}\, dx}{\int_0^2 e^{2x}\, dx}$$

(Observe that the bottom integral is the volume integral.) Computing the bottom integral is straightforward; for the top we use integration by parts with $u = x$ and $dv = e^{2x}\, dx$, so that $du = dx$ and $v = \frac{1}{2}e^{2x}$:

$$\overline{x} = \frac{\left[x \cdot \frac{1}{2}e^{2x}\right]_0^2 - \int_0^2 \frac{1}{2}e^{2x}\, dx}{\left[\frac{1}{2}e^{2x}\right]_0^2} = \frac{\frac{1}{2}\left[xe^{2x} - \frac{1}{2}e^{2x}\right]_0^2}{\frac{1}{2}(e^4 - 1)} = \frac{\frac{3}{2}e^4 + \frac{1}{2}}{e^4 - 1} = \frac{3e^4 + 1}{2(e^4 - 1)}.$$

Half credit (unrotated region): Assume uniform density 1. Applying the center of mass formula directly, we have

$$\overline{x} = \frac{\int_0^2 x \cdot e^x\, dx}{\int_0^2 e^x\, dx} = \frac{e^2 + 1}{e^2 - 1}$$

where the integral of the top was previously computed in part (a).