Math 132
Midterm Examination 3 Solutions – April 6, 2012

6 multiple choice, 4 long answer. 100 points.

Part I was multiple choice. Only the correct answers are listed here.

1. The geometric series $\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{3^i}$ converges to:

   (c) $\frac{1}{4}$

2. Evaluate $\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx$.

   (g) 6

3. Evaluate $\int_4^{\infty} \frac{1}{x^{5/2}}\,dx$.

   (d) $\frac{1}{12}$

4. Evaluate $\int_{-2}^{2} \frac{1}{x^2}\,dx$.

   (j) Does not exist/undefined/diverges.
   (Because $\int_0^1 \frac{1}{x^2}\,dx = \infty$ diverges.)

5. Evaluate $\int_0^{\pi/2} \sin 2\theta \cos\theta \, d\theta$.

   (f) $\frac{2}{3}$

6. The sequence $\frac{n^4}{n!}$ converges (as $n \to \infty$) to:

   (e) 0

Part II was long answer.

1. Exact evaluation of improper integrals and series

   (a) (6 points) Evaluate $\sum_{i=1}^{\infty} \frac{2^i + 3^i}{4^i}$.

   $$= \left(\sum_{i=1}^{\infty} \frac{2^i}{4^i}\right) + \left(\sum_{i=1}^{\infty} \frac{3^i}{4^i}\right) = \frac{2/4}{1-2/4} + \frac{3/4}{1-3/4} = 1 + 3 = 4$$

   (using the geometric series formulas)

   (b) (6 points) For what values of x does $\sum_{i=0}^{\infty} x^i$ converge? When it converges, what does it converge to?

   The series is geometric with ratio x, so it converges for $|x| < 1$. (I.e., for x on $(-1, 1)$). The initial term is 1, so when it converges it converges to $\frac{1}{1-x}$.

   (c) (6 points) Evaluate $\int_0^{\infty} xe^{-x}\,dx$.

   Using integration by parts with $u = x$, $dv = e^{-x}\,dx$ (so that $du = dx$ and $v = -e^{-x}$):

   $$= \left[-xe^{-x}\right]_0^{\infty} + \int_0^{\infty} e^{-x}\,dx = \left[-xe^{-x} - e^{-x}\right]_0^{\infty}$$
   $$= \lim_{t\to\infty} \left(-\frac{t}{e^t} - \frac{1}{e^t} + 0 \cdot e^0 + e^0\right) = 0 + 0 + 0 + 1 = 1.$$

   (d) (6 points) Using partial fractions, evaluate $\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)}$.

   Using partial fractions, we observe that

   $$\frac{1}{(k+1)\cdot(k+2)} = \frac{1}{k+1} - \frac{1}{k+2}.$$

   Then the sum will telescope:

   $$\frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \dots$$

   In general, the partial sum $A_n = \frac{1}{2} - \frac{1}{n+2}$ converges to $\frac{1}{2} - 0$. Hence

   $$\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)} = \frac{1}{2}.$$

2. Integration techniques

(a) (6 points) Evaluate $\int \frac{z+4}{z^3+z}\,dz$.

We factor the bottom: $z^3+z = z\cdot(z^2+1)$. Thus,

$$\frac{z+4}{z^3+z} = \frac{A}{z} + \frac{Bz+C}{z^2+1}, \text{ and so}$$

$$z+4 = A(z^2+1) + (Bz+C)\cdot z.$$

Plugging in 0 gives $0+4 = A\cdot 1$, i.e. $A=4$. Then we have

$$0z^2 + 1z + 4 = (4+B)z^2 + Cz + 4$$

and so $B=-4$, $C=1$. Thus

$$\begin{aligned} \int \frac{z+4}{z^3+z}\,dz &= \int \frac{4}{z} - \frac{4z}{z^2+1} + \frac{1}{z^2+1}\,dz \\ &= 4\ln|z| - 2\ln(z^2+1) + \tan^{-1} z + C. \end{aligned}$$

(b) (6 points) Evaluate $\int \frac{-9x^2-3x+6}{x^4-5x^2+4}\,dx$.

We factor $x^4-5x^2+4 = (x^2-4)(x^2-1) = (x+2)(x-2)(x+1)(x-1)$. Thus,

$$\frac{-9x^2-3x+6}{x^4-5x^2+4} = \frac{A}{x+2} + \frac{B}{x-2} + \frac{C}{x+1} + \frac{D}{x-1}, \text{ and}$$

$$\begin{aligned} -9x^2-3x+6 &= A(x-2)(x+1)(x-1) + B(x+2)(x+1)(x-1) \\ &\quad + C(x+2)(x-2)(x-1) + D(x+2)(x-2)(x+1). \end{aligned}$$

We plug in $-2, 2, -1, 1$ respectively to get $-36+6+6 = A\cdot(-12)$, $-36-6+6 = B\cdot(-12)$, $-9+3+6 = C\cdot(6)$, $-9-3+6 = D\cdot(-6)$. Hence $A=2$, $B=-3$, $C=0$, and $D=1$, and our integral is:

$$\int \frac{2}{x+2} - \frac{3}{x-2} + \frac{1}{x-1}\,dx = 2\ln|x+2| - 3\ln|x-2| + \ln|x-1| + C.$$

(c) (6 points) Evaluate $\int \frac{e^{3x}}{\sqrt{1-e^{2x}}}\,dx$.

First substitute $u=e^x$ (so $du = e^x\,dx$) to get

$$\int \frac{e^{3x}}{\sqrt{1-e^{2x}}}\,dx = \int \frac{u^2}{\sqrt{1-u^2}}\,du,$$

then substitute $u = \sin t$ (so that $du = \cos t\,dt$) to get

$$= \int \frac{\sin^2 t}{\sqrt{1-\sin^2 t}}\cdot \cos t\,dt = \int \frac{\sin^2 t}{\cos t}\cos t\,dt = \int \sin^2 t\,dt$$

We use a double angle formula to finish:

$$= \int \frac{1-\cos 2t}{2}\,dt = \frac{1}{2}t - \frac{\sin 2t}{4} = \frac{1}{2}t - \frac{2\sin t\cos t}{4} = \frac{1}{2}\sin^{-1} e^x - \frac{1}{2}e^x\cdot\sqrt{1-e^{2x}} + C.$$

(There are several equivalent ways of writing this last, and any such equivalent form is fine.)

(d) (5 points) Evaluate $\displaystyle\int_0^1 \frac{1}{(3-x^2)^{3/2}}\,dx$.

We substitute $x = \sqrt{3}\sin u$ (so that $dx = \sqrt{3}\cos u\,du$) to get

$$= \int_0^{\sin^{-1}\frac{1}{\sqrt{3}}} \frac{\sqrt{3}\cos u\,du}{(3-3\sin^2 u)^{3/2}} = \int_0^{\sin^{-1}\frac{1}{\sqrt{3}}} \frac{\sqrt{3}\cos u\,du}{(\sqrt{3}\cos u)^3}$$

$$= \int_0^{\sin^{-1}\frac{1}{\sqrt{3}}} \frac{\sec^2 u}{3}\,du = \left[\frac{\tan u}{3}\right]_0^{\sin^{-1}\frac{1}{\sqrt{3}}} = \frac{1}{3}\frac{\frac{1}{\sqrt{3}}}{\sqrt{1-\frac{1}{3}}} = \frac{1}{3\sqrt{2}}.$$

3. Series convergence.
Determine whether each of the following series converges or diverges.

(a) (6 points) $\displaystyle\sum_{i=1}^{\infty} \frac{2(i-1)(i-2)}{3(i+1)(i+2)}$

Diverges by the Limit Test for Divergence (mth Term Test):

$$\lim_{i\to\infty} a_i = \lim_{i\to\infty} \frac{2(i-1)(i-2)}{3(i+1)(i+2)} = \frac{2}{3} \neq 0.$$

(b) (6 points) $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^{3/2}+\sin^2 i}$

Solution 1: Converges by Limit Comparison with $\frac{1}{i^{3/2}}$:

$$\lim_{i\to\infty} \frac{\frac{1}{i^{3/2}+\sin^2 i}}{\frac{1}{i^{3/2}}} = \lim_{i\to\infty} \frac{i^{3/2}}{i^{3/2}+\sin^2 i} = \lim_{i\to\infty} \frac{1}{1+\frac{\sin^2 i}{i^{3/2}}} = 1,$$

so series is equiconvergent with $\sum \frac{1}{i^{3/2}}$, which converges.

Solution 2: Converges by Direct Comparison with $\frac{1}{i^{3/2}}$: We have $i^{3/2}+\sin^2 i \geq i^{3/2}$, hence $0 \leq \frac{1}{i^{3/2}+\sin^2 i} \leq \frac{1}{i^{3/2}}$. Since $\sum \frac{1}{i^{3/2}}$ converges, so does $\sum \frac{1}{i^{3/2}+\sin^2 i}$.

(c) (6 points) $\displaystyle\sum_{i=1}^{\infty} \frac{2}{i+\ln i}$

Solution 1: Diverges by Limit Comparison with $\frac{1}{i}$:

$$\lim_{i\to\infty} \frac{\frac{2}{i+\ln i}}{\frac{1}{i}} = \lim_{i\to\infty} \frac{i}{i+\ln i} = \lim_{i\to\infty} \frac{1}{1+\frac{\ln i}{i}} = 1,$$

so the series is equiconvergent with $\sum \frac{1}{i}$, which diverges.
Solution 2: Diverges by Direct Comparison with $\frac{1}{i}$: We have $i > \ln i$ (for $i \geq 3$), so $i + \ln i \geq 2i$ and $\frac{2}{i+\ln i} \geq \frac{2}{2i} = \frac{1}{i}$ (all for $i \geq 3$). Since $\sum \frac{1}{i}$ diverges, so does $\sum \frac{1}{i+\ln i}$.

4. Comparison tests for integrals
Use a test for convergence for each problem on this page. (Don't try to find anti-derivatives!)

(a) (5 points) Show that $\displaystyle\int_1^\infty \frac{1}{x^2+\sqrt{x}}\,dx$ converges.

By Direct Comparison: $x^2+\sqrt{x} \ge x^2$, so $0 \le \frac{1}{x^2+\sqrt{x}} \le \frac{1}{x^2}$, and since $\int_1^\infty \frac{1}{x^2}\,dx$ converges, so does $\int_1^\infty \frac{1}{x^2+\sqrt{x}}\,dx$.
(A solution by Limit Comparison with $\frac{1}{x^2}$ is similarly straightforward.)

(b) (5 points) Show that $\displaystyle\int_0^1 \frac{1}{x^2+\sqrt{x}}\,dx$ converges.

By Direct Comparison: $x^2+\sqrt{x} \ge \sqrt{x}$, so $0 \le \frac{1}{x^2+\sqrt{x}} \le \frac{1}{\sqrt{x}}$, and since $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges, so does $\int_0^1 \frac{1}{x^2+\sqrt{x}}\,dx$.

(c) (1 point) Conclude that $\displaystyle\int_0^\infty \frac{1}{x^2+\sqrt{x}}\,dx$ converges.

It converges because

$$\int_0^\infty \frac{1}{x^2+\sqrt{x}}\,dx = \int_0^1 \frac{1}{x^2+\sqrt{x}}\,dx + \int_1^\infty \frac{1}{x^2+\sqrt{x}}\,dx,$$

and because both of the latter integrals converge.