Math 132
Comparing Improper Integrals – April 2, 2012

Theorem 1 (Direct comparison at ∞)**.** *Let f and g be functions with no vertical asymptotes on $[b, \infty)$, and suppose that $0 \le f(x) \le g(x)$ for all x on $[b, \infty)$. Then*

1. *If $\int_b^\infty f(x)\,dx = \infty$, then also $\int_b^\infty g(x)\,dx = \infty$.*
2. *If $\int_b^\infty g(x)\,dx$ converges, then so does $\int_b^\infty f(x)\,dx$.*

We've also referred to this as the Domination Test, since the essential ingredient is domination:

$$\text{if } f(x) \le g(x), \text{ then } \int_b^c f(x)\,dx \le \int_b^c g(x)\,dx.$$

(as discussed at the beginning of the course).

The statement "at" $-\infty$ is entirely similar.

The statements from both sides of a vertical asymptote are also similar, but let us state them carefully:

Theorem 2 (Direct comparison at a, from the right)**.** *Let f and g be functions with an asymptote at a, and no other asymptotes on the interval $[a, b]$, and suppose that $0 \le f(x) \le g(x)$ for all x on $[a, b]$. Then*

1. *If $\int_a^b f(x)\,dx = \infty$, then also $\int_a^b g(x)\,dx = \infty$.*
2. *If $\int_a^b g(x)\,dx$ converges, then so does $\int_a^b f(x)\,dx$.*

Theorem 3 (Direct comparison at a, from the left)**.** *Let f and g be functions with an asymptote at a, and no other asymptotes on the interval $[c, a]$, and suppose that $0 \le f(x) \le g(x)$ for all x on $[c, a]$. Then*

1. *If $\int_c^a f(x)\,dx = \infty$, then also $\int_c^a g(x)\,dx = \infty$.*
2. *If $\int_c^a g(x)\,dx$ converges, then so does $\int_c^a f(x)\,dx$.*