Webwork 4, Problem 13

Find the area of the surface obtained by rotating $x = \frac{1}{3}(y^2+2)^{3/2}$, $1 \le y \le 2$, around the x-axis.

Solution 1: Since we are rotating around the x-axis, we solve for y:

$$3x = (y^2+2)^{3/2}$$
$$(3x)^{2/3} = y^2+2$$
$$\sqrt{(3x)^{2/3}-2} = y = f(x).$$

Our surface area is then

$$2\pi \int_{\sqrt{3}}^{2\sqrt{6}} f(x)\cdot\sqrt{1+(f'(x))^2}\,dx,$$

where $\sqrt{3} = \frac{1}{3}(1^2+2)^{3/2}$, $2\sqrt{6} = \frac{1}{3}(2^2+2)^{3/2}$.

Calculate $f'(x) = \dfrac{\frac{2}{3}(3x)^{-1/3}\cdot 3}{2\sqrt{(3x)^{2/3}-2}} = \dfrac{(3x)^{-1/3}}{\sqrt{(3x)^{2/3}-2}} = \dfrac{(3x)^{-1/3}}{f(x)}.$

The integral becomes:

$$2\pi \int_{\sqrt{3}}^{2\sqrt{6}} f(x)\cdot\sqrt{1+\frac{(3x)^{-2/3}}{f(x)^2}}\,dx = 2\pi\int_{\sqrt{3}}^{2\sqrt{6}} \cancel{f(x)}\cdot\sqrt{\frac{(3x)^{2/3}-2+(3x)^{-2/3}}{\cancel{f(x)^2}}}\,dx$$

Simplify the algebra (cancel)

$$= 2\pi\int_{\sqrt{3}}^{2\sqrt{6}} \sqrt{(3x)^{2/3}-2+(3x)^{-2/3}}\,dx = 2\pi\int_{\sqrt{3}}^{2\sqrt{6}} \sqrt{\left((3x)^{1/3}-(3x)^{-1/3}\right)^2}\,dx$$

where the latter is the square technique from class. Continuing, we have

$$= 2\pi\int_{\sqrt{3}}^{2\sqrt{6}} (3x)^{1/3} - (3x)^{-1/3}\,dx = 2\pi\left[\frac{3}{4}(3x)^{4/3}\cdot\frac{1}{3} - \frac{3}{2}(3x)^{2/3}\cdot\frac{1}{3}\right]$$

$$= 2\pi\left(\frac{3}{4}\cdot 36\cdot\frac{1}{3} - \frac{3}{2}\cdot 6\cdot\frac{1}{3} - \frac{3}{4}\cdot 9\cdot\frac{1}{3} + \frac{3}{2}\cdot 3\cdot\frac{1}{3}\right)$$

(since $(3\sqrt{3})^{1/3} = \sqrt{3} \Rightarrow (3\sqrt{3})^{4/3} = 9$; similarly for other terms).

$$= 10.5\cdot\pi.$$

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Solution 2: We have

$$2\pi \int_{\sqrt{3}}^{2\sqrt{6}} f(x) \cdot \sqrt{1+(f'(x))^2}\, dx$$

and substitute

$$\left.\begin{aligned} y &= f(x) \\ dy &= f'(x) \cdot dx \end{aligned}\right\}$$ essentially, "change variables from x to y."

So

$$SA = 2\pi \int_1^2 y \cdot \sqrt{1+(f'(x))^2}\, \frac{1}{f'(x)}\, dy$$

$$= 2\pi \int_1^2 y \sqrt{\frac{1}{(f'(x))^2} + 1}\, dy$$

We recall the derivative rule for inverses: $\frac{1}{\frac{dy}{dx}} = \frac{dx}{dy}$

so the integral is

$$= 2\pi \int_1^2 y \cdot \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$$

and $x = \frac{1}{3}(y^2+2)^{3/2} \Rightarrow \frac{dx}{dy} = \frac{1}{3} \cdot \frac{3}{2} \cdot (y^2+2)^{1/2} \cdot 2y$

$$= y \cdot \sqrt{y^2+2}$$

and so

$$SA = 2\pi \int_1^2 y \cdot \sqrt{y^2 \cdot (y^2+2)+1}\, dy = 2\pi \int_1^2 y \cdot \sqrt{\underbrace{y^4+2y^2+1}_{(y^2+1)^2}}\, dy$$

$$= 2\pi \int_1^2 y \cdot (y^2+1)\, dy$$

Substitute $u = y^2+1$, $du = 2y\,dy$

$$= \pi \int_2^5 u\, du = \pi \cdot \left[\frac{u^2}{2}\right]_2^5 = \frac{\pi}{2} \cdot (25-4) = \frac{21}{2}\pi$$

(agreeing w/ Solution 1). ✓