Math 233 Fall 2004

HOMEWORK # 10

(Due : Thursday 12/9 . Either in class or Math Office before 5 PM)

1) Use theorems we've learned to show that for any two points A , B in R^3 , $\int_C z^2\, dx + 2y\, dy + 2xz\, dz$ is the same for any curve C connecting A to B .

Solution : By Fundemental Theorem of Line integrals , $\boldsymbol{F}$ *is path independant if* $\boldsymbol{F}$ *is conservative . Then by component Test* $\boldsymbol{F}$ *is conservative if we have a special condition on the partial derivatives of P , Q and R . Let us check that condition on* $\boldsymbol{F} = z^2\,\boldsymbol{i} + 2y\,\boldsymbol{j} + 2xz\,\boldsymbol{k}$.

$\frac{\partial R}{\partial y} = 0 = \frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial z} = 2z = \frac{\partial R}{\partial x}$, $\frac{\partial Q}{\partial x} = 0 = \frac{\partial P}{\partial y}$.

Could also be done by showing $\boldsymbol{F} = \nabla (xz^2 + y^2)$.

2) Find a potential function f(x,y,z) for the vector field $\mathbf{F}(x,y,z) = (e^x\, ln(y))\, \boldsymbol{i} + (\frac{e^x}{y} + sin(z))\, \boldsymbol{j} + (y\, cos(z))\, \boldsymbol{k}$.

(The answer without showing how you got it is not sufficient)

Solution : If $\boldsymbol{F} = \nabla(f)$ *, then* $f_x = e^x ln(y) \rightarrow f = e^x ln(y) + g(y,z)$.

Next $f_y = \frac{e^x}{y} + \frac{\partial g}{\partial y} = \frac{e^x}{y} + sin(z) \rightarrow f = e^x ln(y) + y\, sin(z) + g(z)$.

Next $f_z = y\, cos(z) + g'(z) = y\, cos(z) \rightarrow f = e^x ln(y) + y\, sin(z)$.

3) Let $\mathbf{F} = \nabla (x^3 y^2)$. Calculate $\int_C \mathbf{F} \cdot \mathbf{dr}$, where C is the path connecting $(-1 , 1)$ to $(1 , 1)$ that consists of C_1 , the line segment from $(-1 , 1)$ to $(0 , 0)$, followed by C_2 , the line segment from $(0 , 0)$ to $(1 , 1)$

a) by finding parametrizations for C_1 and C_2 and evaluating integral , and

b) by using the potential vector , $f(x , y) = x^3y^2$.

Solution :

a) C_1 *is* $\boldsymbol{r_1}(t) = < -1 + t , 1 - t >$ *and* $\boldsymbol{r}_1'(t) = < 1, -1 >$.

C_2 *is* $\boldsymbol{r_2}(t) = < t, t >$ *and* $\boldsymbol{r}_2'(t) = < 1, 1 >$. *Now* $\int_C = \int_{C_1} + \int_{C_2}$.

$\boldsymbol{F} = < 3x^2y^2 , 2x^3y >$. *For* C_1 $\boldsymbol{F} = < 3(t-1)^4 , -2(t-1)^4 >$ *and*

$\int_{C_1} \boldsymbol{F} \cdot \boldsymbol{dr_2} = \int_0^1 [3(t-1)^4 + 2(t-1)^4]\, dt = \int_0^1 5(t-1)^4\, dt = 1$.

For C_2 , $\boldsymbol{F} = < 3t^4 , 2t^4 >$ *and* $\int_{C_2} \boldsymbol{F} \cdot \boldsymbol{dr_2} = \int_0^1 5t^4\, dt = 1$.

Then $\int_C \boldsymbol{F} \cdot \boldsymbol{dr} = 2$.

b) *By Fundemental Th*^{m} $\int_C \boldsymbol{F} \cdot \boldsymbol{dr} = f(1,1) - f(-1,1) = 2$

4) For what values of the numbers b and c , will **F** be a gradient vector field , where **F** = $(y^2 + 2czx)\,\boldsymbol{i} + y(bx + cz)\,\boldsymbol{j} + (y^2 + cx^2)\,\boldsymbol{k}$.

Solution : *As in problem* 1, *we will use the component test . If* $\boldsymbol{F} = \nabla(f)$ *then* : $\frac{\partial R}{\partial y} = 2y = \frac{\partial Q}{\partial z} = cy$, *so* **c = 2** . *Next* $\frac{\partial P}{\partial z} = 2cx = \frac{\partial R}{\partial x} = 2cx$, *OK* .

Finally we have $\frac{\partial Q}{\partial x} = by = \frac{\partial P}{\partial y} = 2y$, *hence we get* **b = 2** .

The answer is $b = c = 2$.

5) Verify the statement of Green's Theorem for **F** = $(-x^2y)\,\boldsymbol{i} + (\boldsymbol{xy^2})\,\boldsymbol{k}$, where C is the circle $x^2 + y^2 = a^2$ travelled in the counterclockwise direction from $(a, 0)$ back to $(a, 0)$.

(Note : C can be $\mathbf{r} = (a\cos(t))\,\mathbf{i} + (a\sin(t))\,\mathbf{j}$, $0 \le t \le 2\pi$)

Solution : 1) $\int_C P\,dx + Q\,dy =$

$\int_0^{2\pi} [\, -a^2\cos^2(t)\, a\sin(t)\, (-a\sin(t)) + a\cos(t) a^2 \sin^2(t)\, (a\cos(t))\,]\, dt =$

$\int_0^{2\pi} a^4\cos^2(t)\sin^2(t) + a^4\cos^2(t)\sin^2(t)\, dt = 2a^4 \int_0^{2\pi} \cos^2(t)\sin^2(t)\, dt =$

$2a^4 \int_0^{2\pi} \frac{\sin^2(2t)}{4}\, dt = \frac{a^4}{4}\int_0^{2\pi} 1 - \cos(4t)\, dt = \frac{a^4}{4}\left(t - \frac{\sin(4t)}{4}\right)\Big|_0^{2\pi} = \frac{\pi a^4}{2}$.

2) $\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int\int_D (y^2 + x^2)\, dA = \int_0^{2\pi}\int_0^a r^3\, dr\, d\theta =$

$2\pi\left(\frac{r^4}{4}\Big|_0^a\right) = \frac{\pi a^4}{2}$. *So Green's Theorem is true for* $\boldsymbol{F}$.

6) Apply Green's Theorem to evaluate $\int_C 3y\,dx + 2x\,dy$, where C is a boundary of the region $0 \le x \le \pi$, $0 \le y \le \sin(x)$.

Solution : *By Green's theorem this integral is the same as* $\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$, *where D is the region described above* .

This equals $\int_0^{\pi}\int_0^{\sin(x)} (2 - 3)\, dy\, dx = \int_0^{\pi} -\sin(x)\, dx =$

$\cos(x)\Big|_0^{\pi} = -2$.