Measures on Semi-Rings in $R^1$ and $R^k$

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1. Introduction. The purpose here is to provide an efficient way of deriving Borel measures (in particular Lebesgue-Stieltjes measures in $R^1$ and $R^k$) using semi-rings of subsets of a set $X$. We feel that this is a more efficient and more heuristic approach that using algebras of subsets of $X$, even though using algebras may provide shorter proofs if certain combinatorial lemmas are viewed as obvious.

2. Semi-rings of Sets. In general, a semi-ring of subsets of a set $X$ is a collection $\Gamma$ of subsets of $X$ such that

(i) $\phi \in \Gamma$
(ii) $A, B \in \Gamma$ implies $A \cap B \in \Gamma$
(iii) For any $A, B \in \Gamma$, there exists an integer $m$ and disjoint sets $C_1, \ldots, C_m \in \Gamma$ such that $A - B = \bigcup_{j=1}^{m} C_j$.

Examples: (1) $P(X) = 2^X$, the set of all subsets of $X$.

(2) For $X = R^1$, the set $\Gamma$ of all cells or $h$-intervals $(a, b]$ for $-\infty < a \leq b < \infty$. Another example is the slightly larger collection $\Gamma_1$ of cells with $-\infty \leq a \leq b \leq \infty$. Note that the condition $a = b$ allows $\phi \in \Gamma$.

(3) For $X = R^k$, the set $\Gamma$ of all cells $\prod_{j=1}^{k} (a_j, b_j]$ where $\prod$ denotes the Cartesian product and $-\infty < a_j \leq b_j < \infty$. As in Example (2), we can also allow $a_j = -\infty$ and $b_j = \infty$.

(4) Any $\sigma$-algebra $\mathcal{M}$ of subsets of $X$. Recall that $\mathcal{M}$ is a $\sigma$-algebra of subsets of $X$ if

(i) $\phi \in \mathcal{M}$
(ii) $A \in \mathcal{M}$ implies $A^c \in \mathcal{M}$
(iii) If $A_j \in \mathcal{M}$ for $1 \leq j < \infty$, then $A = \bigcup_{j=1}^{\infty} A_j \in \mathcal{M}$.

Exercise: Verify that Examples (2) and (3) are semi-rings, and that we can take $m \leq 2$ in part (iii) for Example (2) and $m \leq 2^k$ in Example (3).
Definition: A set function $\mu(A)$ is a premeasure on a semi-ring $\Gamma$ if $\mu : \Gamma \to [0, \infty]$ is a function such that

(i) $\mu(\emptyset) = 0$

(ii) If $A_k \in \Gamma$ are disjoint for $1 \leq k < \infty$, and if $A = \bigcup_{k=1}^{\infty} A_k \in \Gamma$, then $\mu(A) = \sum_{k=1}^{\infty} \mu(A_k)$.

A premeasure $\mu_0$ is $\sigma$-finite if $X = \bigcup_{j=1}^{\infty} X_j$ where $X_j \in \Gamma$ and $\mu_0(X_j) < \infty$.

Notes: (a) In particular, $\mu(A) = \infty$ for $A \in \Gamma$ is allowed.

(b) $\mu(A)$ is also finitely-additive on $\Gamma$. That is, if $A, A_j \in \Gamma$, $A_j$ is disjoint, and $A = \bigcup_{j=1}^{n} A_j$ satisfies $A \in \Gamma$, then $\mu(A) = \sum_{j=1}^{n} \mu(A_j)$. This is because we can take $A_j = \emptyset$ for $j > n$ in property (ii) above. If $\mu(A)$ is only finitely
additive; that is, if (ii) is only guaranteed if $A_j = \emptyset$ for $j > n$ for some finite $n$, then we call $\mu(A)$ a finitely-additive premeasure on $\Gamma$.

(c) If $A, B \in \Gamma$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$ by property (iii) of the definition of a semi-ring, property (ii) of the definition of a premeasure, and the property that $\mu(C) \geq 0$ for $C \in \Gamma$. Thus a premeasure (or finitely-additive premeasure) on a semi-ring is automatically monotone.

Definition: $\mu(A)$ is a measure on a $\sigma$-algebra $\mathcal{M}$ if $\mu : \mathcal{M} \to [0, \infty]$ satisfies

(i) $\mu(\emptyset) = 0$

(ii) If $A_k \in \mathcal{M}$ are disjoint for $1 \leq k < \infty$ and $A = \bigcup_{k=1}^{\infty} A_k$ (which is automatically in $\mathcal{M}$), then $\mu(A) = \sum_{k=1}^{\infty} \mu(A_k)$.

The following three lemmas are useful for working with semi-rings.

Lemma 2.1. Assume sets $A, A_1, \ldots, A_n \in \Gamma$ for a semi-ring $\Gamma$. Then there exists $m < \infty$ and disjoint sets $D_1, D_2, \ldots, D_m \in \Gamma$ such that

$$A - \bigcup_{j=1}^{n} A_j = A - A_1 - A_2 - \cdots - A_n = \bigcup_{k=1}^{m} D_k \quad (2.1)$$

Proof. By condition (iii) for semi-rings, $A - A_1 = \bigcup_{j=1}^{n} C_j$ for disjoint $C_j \in \Gamma$. Then $A - A_1 - A_2 = \bigcup_{j=1}^{n} C_j - A_2 = \bigcup_{j=1}^{m} (C_j - A_2) = \bigcup_{j=1}^{m} \bigcup_{k=1}^{n_j} D_{jk}$ where $D_{jk} \in \Gamma$ are disjoint for fixed $j$ with $\bigcup_{k=1}^{n_j} D_{jk} = C_j - A_2$. Since the $C_j$ are disjoint with $D_{jk} \subseteq C_j$, the $D_{jk}$ are disjoint for all $j, k$. Thus we can write $A - A_1 - A_2 = \bigcup_{k=1}^{M} \tilde{D}_k$ for disjoint $\tilde{D}_k \in \Gamma$ and $M \leq n_1 + \ldots + n_m$. Lemma 2.1 for all $n$ follows by induction on $n$.

Exercise: Show that we can take $m \leq 2^n$ for the semi-ring of cells $\Gamma$ in Example (2). For cells in $R^k$ (Example (3)), we can take $m \leq 2^{nk}$. 

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**Lemma 2.2.** Let $\mu(A)$ be a finitely-additive premeasure on a semi-ring $\Gamma$. Assume $A, A_1, \ldots, A_n \in \Gamma$ are such that $A_1, \ldots, A_n$ are disjoint and $\bigcup_{j=1}^{n} A_j \subseteq A$. Then

$$\sum_{j=1}^{n} \mu(A_j) \leq \mu(A) \quad (2.2)$$

**Proof.** By Lemma 2.1, $A - \bigcup_{j=1}^{n} A_j = \bigcup_{k=1}^{m} D_k$ where $D_k \in \Gamma$ are disjoint, also disjoint from $A_1, \ldots, A_n$. Thus $\{A_1, \ldots, A_n, D_1, \ldots, D_m\}$ are disjoint and by finite additivity

$$\mu(A) = \sum_{j=1}^{n} \mu(A_j) + \sum_{k=1}^{m} \mu(D_k) \geq \sum_{j=1}^{n} \mu(A_j)$$

since $\mu(D_k) \geq 0$.

**Lemma 2.3.** Let $\mu(A)$ be a finitely-additive premeasure on a semi-ring $\Gamma$. Assume $A, A_1, \ldots, A_n \in \Gamma$ are such that $A \subseteq \bigcup_{j=1}^{n} A_j$. Then

$$\mu(A) \leq \sum_{j=1}^{n} \mu(A_j) \quad (2.3)$$

**Proof.** Since $A \subseteq \bigcup_{j=1}^{n} A_j$,

$$A = A \cap \bigcup_{j=1}^{n} A_j = \bigcup_{j=1}^{n} (A \cap A_j) = \bigcup_{j=1}^{n} \tilde{A}_j, \quad \tilde{A}_j = (A \cap A_j) - \bigcup_{k=1}^{j-1} (A \cap A_k)$$

Each $A \cap A_k \in \Gamma$ by condition (ii) of the definition of a semi-ring. The sets $\tilde{A}_j$ are disjoint, but are not necessarily in $\Gamma$. By Lemma 2.1, each $A_j = \bigcup_{k=1}^{n_j} D_{jk}$ where $D_{jk} \in \Gamma$ are disjoint for fixed $j$. Since the $\tilde{A}_j$ are disjoint, the sets $D_{jk} \in \Gamma$ are disjoint for all $j, k$. Since $\mu$ is finitely additive,

$$\mu(A) = \sum_{j=1}^{n} \sum_{k=1}^{n_j} \mu(D_{ij}) \leq \sum_{j=1}^{n} \mu(A_j)$$

by Lemma 2.2 since $\bigcup_{k=1}^{n_j} D_{jk} = \tilde{A}_j \subseteq A_j$. 
3. Semi-rings and Outer Measures. An outer measure on a set $X$ is a function $\mu^* : P(X) \to [0, \infty]$ where $P(X)$ is the set of all subsets $E \subseteq X$ such that $\mu^*$ satisfies

(i) $\mu^*(\emptyset) = 0$

(ii) $E \subseteq F \subseteq X$ implies $\mu^*(E) \leq \mu^*(F)$

(iii) If $E_j \subseteq X$ for $1 \leq j < \infty$ and $E = \bigcup_{j=1}^{\infty} E_j$, then $\mu^*(E) \leq \sum_{j=1}^{\infty} \mu^*(E_j)$.

Note that outer measures are defined for all subsets $E$ of a set $X$ rather than on a semi-ring or $\sigma$-algebra.

Definition: A set $A \subseteq X$ is $\mu^*$-measurable if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$  \hspace{1cm} (3.1)

for all subsets $E \subseteq X$. Define $\mathcal{M}(\mu^*)$ as the set of all $\mu^*$-measurable subsets $A \subseteq X$.

In particular, $A = \emptyset \in \mathcal{M}(\mu^*)$ since (3.1) holds for all $E \subseteq X$. Similarly, $A \in \mathcal{M}(\mu^*)$ implies $A^c \in \mathcal{M}(\mu^*)$, which are two of the three properties required for a $\sigma$-algebra. More generally:

Theorem 3.1 (Carathéodory) Let $\mu^*$ be an arbitrary outer measure on a set $X$. Then

(i) $\mathcal{M}(\mu^*)$ is a $\sigma$-algebra of subsets of $X$

(ii) $\mu^*$ is a (countably-additive) measure on $\mathcal{M}(\mu^*)$.

Proof. See Folland (1999) in the references, or any textbook on measure theory. (This proof does not use semi-rings or algebras of sets.)

Notes: (1) Theorem 3.1 does not guarantee that the $\sigma$-algebra is very large or very interesting. Problem 4 on Homework 1 of Math 5051 (Fall 2009) gives an example of an outer measure $\mu^*$ on $X = [0, 1]$ with $\mu^*(E) > 0$ for all nonempty $E \subseteq [0, 1]$ but $\mathcal{M}(\mu^*) = \{ \emptyset, X \}$.

(2) Let $\mathcal{E} \subseteq P(X)$ be an arbitrary collection of subsets of a set $X$ and let $\mu_0(A)$ be an arbitrary nonnegative function on $\mathcal{E}$. Then

$$\mu^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu_0(A_j) : E \subseteq \bigcup_{j=1}^{\infty} A_j, A_j \in \mathcal{E} \right\}$$ \hspace{1cm} (3.2)

defines an outer measure on $X$. We define $\mu^*(E)$ with the convention that the infimum of the empty set is $\infty$. That is, if $E$ cannot be covered by a sequence of sets $A_j \in \mathcal{E}$ as in (3.2), then $\mu^*(E) = \infty$. (Proof: See Folland (1999).)
Definition: $H$ is a $(\mu^*)$-null set if $H \subseteq X$ and $\mu^*(H) = 0$. If $H$ is a $\mu^*$-null set, then $H \in \mathcal{M}(\mu^*)$. That is, $\mathcal{M}(\mu^*)$ contains all null sets for $\mu_0$. (Proof: If $\mu^*(H) = 0$, then
\[ \mu^*(E) \leq \mu^*(E \cap H) + \mu^*(E \cap H^c) \leq \mu^*(E \cap H^c) \leq \mu^*(E) \]
and (3.1) holds for all $E \subseteq X$. Hence $H \in \mathcal{M}(\mu^*)$.)

The next result shows how to extend an arbitrary premeasure on a semi-ring to a measure on a $\sigma$-algebra.

Theorem 3.2 (Carathéodory) Let $\mu_0$ be a (countably-additive) premeasure on a semi-ring $\Gamma$ of subsets of a set $X$. Define $\mu^*(E)$ by (3.2) for $E = \Gamma$. Then
\begin{itemize}
  \item[(i)] $\mu^*(A) = \mu_0(A)$ for all $A \in \Gamma$
  \item[(ii)] $\Gamma \subseteq \mathcal{M}(\mu^*)$.
\end{itemize}

Notes: (1) For $\mu^*(E)$ as in Theorem 3.2, if we define $\mu(A) = \mu^*(A)$ for $A \in \mathcal{M}(\mu^*)$, then $\mu$ is a measure on both $\mathcal{M}(\mu^*)$ and on the smallest $\sigma$-algebra $\mathcal{M}(\Gamma)$ containing $\Gamma$.

(2) Under the conditions of Theorem 3.2, if $\mu_0$ is $\sigma$-finite on $X$, then every $E \in \mathcal{M}(\mu^*)$ can be written $E = B - H$ where $B \in \mathcal{M}(\Gamma)$ and $\mu^*(H) = 0$. That is, $\mathcal{M}(\mu^*)$ differs from $\mathcal{M}(\Gamma)$ only by null sets. (See Problem 2 on Homework 2 for Math 5051, Fall 2009.)

Proof of Theorem 3.2 (Carathéodory). (i) We first show that $\mu^*(A) = \mu_0(A)$ for any $A \in \Gamma$. Since $A$ is a covering of itself, $\mu^*(A) \leq \mu_0(A)$. Thus it is sufficient to prove $\mu_0(A) \leq \mu^*(A)$.

(Remark: Problem 5 of Homework 2 in Math 5051 (Fall 2009) gives an example of an outer measure defined by (3.2) with $\mu_0(A) > 0$ for every nonempty $A \in \Gamma$ but $\mu^*(E) = 0$ for all sets $E \subseteq X$. Thus some argument is required.)

Given $A \in \Gamma$ with $\mu^*(A) < \infty$ (otherwise $\mu_0(A) \leq \mu^*(A)$ is trivial), choose $A_i \in \Gamma$ such that
\[ A \subseteq \bigcup_{j=1}^{\infty} A_j, \quad \mu^*(A) \leq \sum_{j=1}^{\infty} \mu_0(A_j) \leq \mu^*(A) + \epsilon \]
As in the proof of Lemma 2.3, we can find disjoint $B_k \in \Gamma$ such that
\[ A \subseteq \bigcup_{j=1}^{\infty} A_j = \bigcup_{k=1}^{\infty} B_k, \quad \mu^*(A) \leq \sum_{k=1}^{\infty} \mu_0(B_k) \leq \sum_{j=1}^{\infty} \mu_0(A_j) \]
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Then \( A = \bigcup_{k=1}^{\infty} (A \cap B_k) \) for disjoint sets \( A \cap B_k \in \Gamma \). Thus 

\[
\mu_0(A) = \sum_{k=1}^{\infty} \mu_0(A \cap B_k) \leq \sum_{k=1}^{\infty} \mu_0(B_k) \leq \sum_{j=1}^{\infty} \mu_0(A_j) \leq \mu^*(A) + \epsilon
\]

since \( \mu_0(A) \leq \mu_0(B) \) if \( A \subseteq B, A, B \in \Gamma \). This implies \( \mu_0(A) \leq \mu^*(A) \) and hence \( \mu_0(A) = \mu^*(A) \).

(ii) We next show that any \( A \in \Gamma \) satisfies \( A \in \mathcal{M}(\mu^*) \). Since \( \mu^* \) is subadditive (that is, property (iii) of the definition of outer measure), it is sufficient to prove 

\[
\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)
\]

for all subsets \( E \subseteq X \). Choose \( A_j \in \Gamma \) such that 

\[
E \subseteq \bigcup_{j=1}^{\infty} A_j, \quad \mu^*(E) \leq \sum_{j=1}^{\infty} \mu_0(A_j) \leq \mu^*(E) + \epsilon
\]

By property (iii) of the definition of a semi-ring 

\[
E \cap A \subseteq \bigcup_{j=1}^{\infty} (A \cap A_j), \quad E \cap A^c \subseteq \bigcup_{j=1}^{\infty} (A_j - A) = \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{n_j} D_{jk}
\]

where \( A_j - A = \bigcup_{k=1}^{n_j} D_{jk} \) for disjoint \( D_{jk} \in \Gamma \). Thus 

\[
\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \sum_{j=1}^{\infty} \mu_0(A \cap A_j) + \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \mu_0(D_{jk}) \\
= \sum_{j=1}^{\infty} \left( \mu_0(A \cap A_j) + \sum_{k=1}^{n_j} \mu_0(D_{jk}) \right) = \sum_{j=1}^{\infty} \mu_0(A_j) \leq \mu^*(E) + \epsilon
\]

since \( A_j = (A \cap A_j) \cup (A_j - A) \) and \( \mu_0 \) is finitely additive. Thus \( \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c) \) for all \( E \subseteq X \), which completes the proof of the theorem.

An important corollary of Theorem 3.2 is 

**Theorem 3.3 (Fréchet)** Let \( \mu \) and \( \nu \) be two measures on a \( \sigma \)-algebra \( \mathcal{M} \) on a set \( X \). Assume \( \Gamma \subseteq \mathcal{M} \) for a semi-ring \( \Gamma \), that \( \mu(A) = \nu(A) \) for every \( A \in \Gamma \), and that \( \mu \) an \( \nu \) are both \( \sigma \)-finite on \( \Gamma \). Then \( \mu(E) = \nu(E) \) for all \( E \in \mathcal{M} \).
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**Proof.** This is essentially Theorem 1.14 in Folland (1999).

Define $\mu^*(E)$ and $\nu^*(E)$ by (3.2) for $E = \Gamma$. Then $\mu^*(E) = \nu^*(E)$ for all $E \subseteq X$ since $\mu(A) = \nu(A)$ for $A \in \Gamma$. Since measures are countably subadditive, $\mu(E) \leq \mu^*(E)$ and $\nu(E) \leq \nu^*(E)$ for all $E \in \mathcal{M}$. If we can show that $\mu(E) = \mu^*(E)$ for any $E \in \mathcal{M}$ and $\sigma$-finite premeasure $\mu$ on $\Gamma$, then we could conclude $\mu(E) = \mu^*(E) = \nu^*(E) = \nu(E)$ for all $E \in \mathcal{M}(\Gamma)$ and we would be done.

First, assume $\mu^*(E) < \infty$. As in the proofs of Lemma 2.3 and Theorem 3.3, there exist disjoint sets $A_j \in \Gamma$ such that

$$E \subseteq A = \bigcup_{j=1}^{\infty} A_j, \quad \mu^*(E) \leq \sum_{j=1}^{\infty} \mu(A_j) = \mu(A) \leq \mu^*(E) + \epsilon$$

Since the $A_j$ are disjoint and $\mu^*(A_j) = \mu(A_j)$ by Theorem 3.3, $\mu^*(A) = \mu(A)$. Since $E \subseteq A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$, $\mu^*(A - E) \leq \epsilon$. Thus also $\mu(A - E) \leq \mu^*(A - E) \leq \epsilon$ and

$$\mu^*(E) \leq \mu^*(A) = \mu(A) = \mu(E) + \mu(A - E) \leq \mu(E) + \epsilon$$

Thus $\mu^*(E) \leq \mu(E)$ and hence $\mu^*(E) = \mu(E)$ for $\mu^*(E) < \infty$.

Since $\mu$ is $\sigma$-finite, $X = \bigcup_{j=1}^{\infty} X_j$ where $X_j \in \Gamma$, $\mu(X_j) < \infty$, and $X_j$ are disjoint. Then for all $E \in \mathcal{M}(\Gamma)$

$$\mu(E) = \sum_{j=1}^{\infty} \mu(E \cap X_j) = \sum_{j=1}^{\infty} \mu^*(E \cap X_j) = \mu^*(E)$$

and $\mu = \mu^*$ on $\mathcal{M}(\Gamma)$, which was to be proven.

4. **Lebesgue-Stieltjes Measures in $R^1$.** Let $F(x)$ be an increasing real-valued right-continuous function on $R^1$. Let $\Gamma$ be the semi-ring

$$\Gamma = \{ (a, b) : -\infty < a \leq b < \infty \} \quad (4.1)$$

Define $\mu_F$ on $\Gamma$ by

$$\mu_F((a, b]) = F(b) - F(a) \quad (4.2)$$

Then
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**Theorem 4.1.** $\mu_F$ in (4.2) is a premeasure on the semi-ring $\Gamma$. In particular, $\mu_F$ in (4.2) extends to a unique Borel measure on $R^1$.

**Proof.** Since $F(x)$ is real-valued, $\mu_F((n,n+1]) < \infty$ and $\mu_F$ is $\sigma$-finite. Once we prove that $\mu_F$ is a premeasure on $\Gamma$, it follows from Theorems 3.3 and 3.4 that $\mu_F$ has a unique extension as a Borel measure on $\mathcal{M}(R^1)$. This will be the Lebesgue-Stieltjes measure on $R^1$ corresponding to $F(x)$. In particular, it is sufficient to prove that $\mu_F$ is a premeasure.

Assume $A = (a,b]$ and $A_j = (a_j,b_j]$ satisfy

$$(a,b] = \bigcup_{j=1}^{\infty} (a_j,b_j] \quad \text{where } (a_j,b_j] \text{ are disjoint}$$

By Lemma 2.2, $\sum_{j=1}^{n} (a_j,b_j] \subseteq (a,b]$ implies

$$\sum_{j=1}^{n} (F(b_j) - F(a_j)) = \sum_{j=1}^{n} \mu_F(A_j) \leq \mu_F(A) = F(b) - F(a)$$

for all $n$. Hence

$$\sum_{j=1}^{\infty} (F(b_j) - F(a_j)) \leq F(b) - F(a) \quad (4.3)$$

Thus it is sufficient to prove

$$F(b) - F(a) \leq \sum_{j=1}^{\infty} (F(b_j) - F(a_j)) \quad (4.4)$$

For any $\epsilon > 0$, there exist $\delta > 0$ and $\delta_j > 0$ such that

$$F(a + \delta) - F(a) < \epsilon \quad \text{and} \quad F(b_j + \delta_j) - F(b_j) < \epsilon/2^j \quad (4.5)$$

for all $j \geq 1$. Then

$$[a + \delta, b] \subseteq (a,b] = \bigcup_{j=1}^{\infty} (a_j,b_j] \subseteq \bigcup_{j=1}^{\infty} (a_j,b_j + \delta_j)$$

Since $[a + \delta, b]$ is compact, it follows that

$$[a + \delta, b] \subseteq \bigcup_{j=1}^{n} (a_j,b_j + \delta_j)$$
for some \( n < \infty \). By Lemma 2.3 and (4.5)

\[
\mu_F((a, b]) = F(b) - F(a) \leq (F(b) - F(a + \delta)) + \epsilon
\]

\[
\leq \sum_{j=1}^{n} (F(b_j + \delta_j) - F(a_j)) + \epsilon
\]

\[
\leq \sum_{j=1}^{n} (F(b_j) - F(a_j)) + \sum_{j=1}^{n} \epsilon/2^j + \epsilon
\]

\[
\leq \sum_{j=1}^{\infty} (F(b_j) - F(a_j)) + 2\epsilon
\]

Since this holds for all \( \epsilon > 0 \), we conclude (4.4) and hence that \( \mu_F \) is a premeasure on \( \Gamma \) in (4.1).

5. Lebesgue-Stieltjes Measures in \( R^k \). Let \( \Gamma \) be the semi-ring

\[
\Gamma = \left\{ C : C = \prod_{j=1}^{k} (a_j, b_j] \text{ for } -\infty < a_j \leq b_j < \infty, \quad 1 \leq j \leq k \right\} \quad (5.1)
\]

where \( \prod_{j=1}^{k} (a_j, b_j] \) means Cartesian product. As in Section 2 (Example 3), \( \Gamma \) is a semi-ring of subsets of \( R^k \). If \( \mu \) is a Borel measure on \( R^k \) and \( \mu(R^k) < \infty \), an analog for \( R^k \) of the increasing function \( F(x) \) in Section 4 is

\[
F(x_1, x_2, \ldots, x_k) = \mu\left( \prod_{j=1}^{k} (-\infty, x_j] \right) \quad (5.2)
\]

We now want the analog of \( \mu((a, b]) = F(b) - F(a) \) in Section 4. Consider the special case of the product measure

\[
\mu = \mu_1 \otimes \mu_2 \otimes \ldots \otimes \mu_k
\]

for one-dimensional measures \( \mu_j \). This means

\[
F(x_1, x_2, \ldots, x_k) = F_1(x_1)F_2(x_2)\ldots F_k(x_k)
\]

where \( F_j(x) = \mu_j\left( (-\infty, x] \right) \) and

\[
\mu\left( \prod_{j=1}^{k} (a_j, b_j] \right) = \prod_{j=1}^{k} (F_j(b_j) - F_j(a_j)) \quad (5.3)
\]
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In particular, if $C = (a, b] \times (c, d] \subseteq R^2$, then

$$
\mu((a, b] \times (c, d]) = (F_1(b) - F_1(a))(F_2(d) - F_2(c)) = F_1(b)F_2(d) - F_1(a)F_2(d) - F_1(b)F_2(c) + F_1(a)F_2(c)
$$

If $F(x_1, x_2)$ in (5.2) is not a product, the generalization is the addition and subtraction formula

$$
\mu((a, b] \times (c, d]) = F(b, d) - F(a, d) - F(b, c) + F(a, c)
$$

(Hint: Draw a picture in the plane.) For general $k$, the expansion of the product (5.3) is a sum with $2^k$ terms:

$$
\mu\left(\prod_{j=1}^{k}(a_j, b_j)\right) = \sum_{c:\{1, \ldots, k\}\rightarrow R} (-1)^{n_c} \prod_{j=1}^{k} F_j(c_j)
$$

If $F(x_1, x_2, \ldots, x_k)$ in (5.2) is not a product, the generalization of (5.5) is

**Lemma 5.1** Let $C = \prod_{j=1}^{k}(a_j, b_j)$. Define $F(x_1, \ldots, x_k)$ by (5.2) where $\mu$ is a Borel measure on $R^k$ with $\mu(R^k) < \infty$. Then

$$
\Delta_C(F) = \mu\left(\prod_{j=1}^{k}(a_j, b_j)\right) = \sum_{c:\{1, \ldots, k\}\rightarrow R} (-1)^{n_c} F(c_1, c_2, \ldots, c_k)
$$

**Proof.** By (5.4) and induction on $k$.

In particular, $F(x_1, x_2, \ldots, x_k)$ in (5.2) satisfies $\Delta_C(F) \geq 0$ for all cells $C \in \Gamma$ in (5.1). This is called the box condition for the function $F(x_1, x_2, \ldots, x_k)$.

The analog of right continuity for $F(x)$ for $x \in R^1$ is the following. We say that $x^n \downarrow x$ for $x^n = (x_1^n, x_2^n, \ldots, x_k^n)$ and $x = (x_1, x_2, \ldots, x_k) \in R^k$ if $x^n_j \downarrow x_j$ for each $j, 1 \leq j \leq k$.

A function $F(x_1, x_2, \ldots, x_k)$ on $R^k$ is jointly right continuous on $R^k$ if $x^n \downarrow x \in R^k$ implies $F(x^n) \rightarrow F(x)$. If the box condition $\Delta_C(F) \geq 0$ holds for all $C \in \Gamma$, then $F(x^n) \downarrow F(x)$.

**Exercises:**

(1) Show that $F(x_1, \ldots, x_n)$ defined by (5.2) is jointly right continuous.

(2) If $F(x_1, \ldots, x_k)$ in (5.2) satisfies the box condition $\Delta_C(F) \geq 0$ for all $C \in \Gamma$, then $x^n \downarrow x$ implies $F(x^n) \downarrow F(x)$.
Theorem 5.1 Let $F : R^k \to R$ be a function such that

(i) For all cells $C \in \Gamma$ in (5.1),

$$\Delta_C(F) = \sum_{c : \{1, \ldots, k\} \to R} (-1)^n c \cdot F(c_1, c_2, \ldots, c_k) \geq 0 \quad (5.7)$$

(ii) $F(x_1, x_2, \ldots, x_k)$ is jointly right continuous.

Then $\mu(C) = \Delta_C(F)$ defined by (5.7) is a premeasure on the semi-ring $\Gamma$ in (5.1).

Note: Then, by the results in Section 3, $\mu(C)$ extends to a unique Borel measure $\mu(A)$ on $R^k$.

Proof of Theorem 5.1. It is not difficult to show (but with some work) that $\mu(C)$ is finitely additive on $\Gamma$ by generalizing the proof that $\mu(C)$ defined by the product measure (5.3) is finitely additive on $\Gamma$. The results in Sections 2 and 3 carry over since they are about general semi-rings and outer measures. Now assume

$$C = \bigcup_{i=1}^{\infty} C_i, \quad C, C_i \in \Gamma, \quad C_i \text{ disjoint}$$

Assume $C = \prod_{j=1}^{k}(a_j, b_j]$ and $C_i = \prod_{j=1}^{k}(a_{ij}, b_{ij}]$. Then

$$\mu(C) \geq \sum_{i=1}^{\infty} \mu(C_i)$$

follows from Lemma 2.2 as in the proof of Theorem 4.1. Define $C_\delta = \prod_{j=1}^{k}(a_j + \delta, b_j]$ and $C_i^{\delta} = \prod_{j=1}^{k}(a_{ij}, b_{ij} + \delta]$. Then, by condition (ii) in Theorem 5.1, for all $\epsilon > 0$, there exist $\delta > 0$ and $\delta_i > 0$ such that

$$\mu(C_\delta) - \mu(C) < \epsilon, \quad \mu(C_i^{\delta_i}) - \mu(C_i) < \epsilon/2^i$$

for $1 \leq i < \infty$. This is the analog of (4.5) in the proof of Theorem 4.1. The rest of the proof of Theorem 4.1 carries over with changes only in the notation. Hence

$$\mu(C) = \sum_{i=1}^{\infty} \mu(C_i)$$

and $\mu(C)$ is a premeasure on $\Gamma$.

References.