

## Math450 HW1 Answers — Spring 2009

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1. (a) Exercise 1 and (b) Exercise 3 at the end of Section 2.1, page 21 on the textbook. (Verify that two formulas were solutions of particular partial differential equations. The verifications are not difficult.)

2. (Solve  $u_t + cu_x = xt$ ,  $u(x, 0) = f(x)$ ) (An inhomogeneous first-order partial differential equation.) Solving first-order PDEs, both inhomogeneous or homogeneous, can be broken into three steps:

**Step 1:** Find the characteristics  $x(t) = x(t, x_0)$  in terms of  $t$  and the initial point  $x(0) = x_0$ . In this case, the characteristic equation is  $(d/dt)x = c$  with  $x(0) = x_0$ , so that the characteristics are

$$x(t) = x(t, x_0) = x_0 + ct \quad (1)$$

The usefulness of characteristics is that

$$(d/dt)u(x(t), t) = (dx/dt)u_x + u_t = u_t + cu_x = x(t)t$$

where the right-hand side that does not depend on any derivatives, so that finding  $v(t) = u(x(t), t)$  along characteristics reduces to, at worst, solving a second first-order ordinary differential equation.

**Step 2:** Find  $u(x, t)$  along each characteristic. Here

$$(d/dt)u(x(t), t) = x(t)t = (x_0 + ct)t = x_0t + ct^2$$

or, if  $v(t) = u(x(t), t)$ , then  $(d/dt)v(t) = x_0t + ct^2$ . Thus

$$v(t) = u(x(t), t) = (1/2)x_0t^2 + (1/3)ct^3 + v(0)$$

Here  $v(0) = u(x(0), 0) = u(x_0, 0) = f(x_0)$ , so that

$$u(x(t), t) = u(x_0 + ct, t) = (1/2)x_0t^2 + (1/3)ct^3 + f(x_0)$$

**Step 3:** Change coordinates from  $(x_0, t)$  back to  $(x, t)$  to derive a formula for  $u(x, t)$  (without reference to characteristics). This can be done by finding the inverse  $p(x, t)$  of  $x(t, x_0)$  for fixed  $t > 0$  so that

$$x = x(t, x_0) \quad \text{if and only if} \quad x_0 = p(x, t)$$

In this case,  $x = x(t, x_0) = x_0 + ct$  if and only if  $x_0 = p(x, t)$  for  $p(x, t) = x - ct$ , so that the solution is

$$\begin{aligned} u(x, t) &= (1/2)(x - ct)t^2 + (1/3)ct^3 + f(x - ct) \\ &= (1/2)xt^2 - (1/6)ct^3 + f(x - ct) \end{aligned}$$

3. (Solve  $u_t + \frac{1}{x}u_x = 0$ ,  $u(x, 0) = f(x)$ , for  $x > 0$ ) Following the three steps of Problem 2,

**Step 1:** Here the characteristics satisfy  $(d/dt)x = 1/x$ , or  $x(d/dt)x = (1/2)(d/dt)x^2 = 1$ , so that  $x(t, x_0)^2 = 2t + 2C = 2t + x_0^2$ , so that the characteristic with  $x(0) = x_0$  is

$$x(t, x_0) = \sqrt{x_0^2 + 2t}$$

**Step 2:** Along characteristics,  $(d/dt)u(x(t), t) = 0$ , so that

$$u(x(t, x_0), t) = u(x_0, 0) = f(x_0)$$

**Step 3:** Change coordinates from  $(x_0, t)$  back to  $(x, t)$  to derive a formula for  $u(x, t)$  without reference to characteristics. If

$$x = x(t, x_0) \quad \text{if and only if} \quad x_0 = p(x, t) \tag{2}$$

then  $u(x, t) = f(p(x, t))$  in this case (and for many homogeneous first-order PDEs in general). Here  $x = \sqrt{x_0^2 + 2t}$  so that  $x^2 = x_0^2 + 2t$  and  $x_0 = p(x, t) = \sqrt{x^2 - 2t}$ . Thus

$$u(x, t) = f(\sqrt{x^2 - 2t})$$

for  $x > 0$  and  $0 \leq t < (1/2)x^2$ .

**4.** (Solve  $u_t + cu_x = u^2$ ,  $u(x, 0) = f(x)$ ) Following the same three steps,

**Step 1:** As in Problem 2, the characteristics are  $x(t, x_0) = x_0 + ct$ .

**Step 2:**  $v(t) = u(x(t), t)$  satisfies

$$(d/dt)v(t) = u_t + cu_x = u(x(t), t)^2 = v(t)^2$$

or  $v(t)^{-2}(d/dt)v(t) = -(d/dt)(1/v(t)) = 1$ . Thus  $-1/v(t) = t + C$  or

$$v(t) = u(x(t, x_0), t) = -1/(t + C) = 1/(C_1 - t), \quad C_1 = -C$$

Thus

$$v(0) = u(x_0, 0) = f(x_0) = 1/C_1 \quad \text{and so} \quad C_1 = 1/f(x_0)$$

which leads to

$$u(x(t, x_0), t) = 1/(C_1 - t) = \frac{f(x_0)}{1 - tf(x_0)}$$

**Step 3:** As in Problem 2,  $x_0 = p(x, t) = x - ct$  if and only if  $x = x(t, x_0) = x_0 + ct$ , so that the solution is

$$u(x, t) = \frac{f(p(x, t))}{1 - tf(p(x, t))} = \frac{f(x - ct)}{1 - tf(x - ct)}$$

which is valid as long as  $tf(x - ct) < 1$  along  $y(t) = x - ct$  (and otherwise blows up).

**5.** (Show that  $u_t + uu_x = 0$ ,  $u(x, 0) = f(x)$  cannot have a continuously-differentiable solution for all  $t > 0$  if there exist  $x_1 < x_2$  such that also  $f(x_2) < f(x_1)$ .) Following the recipe above,

**Step 1:** The characteristics satisfy  $(d/dt)x(t, x_0) = u(x(t), t)$ . However, along a characteristic,

$$(d/dt)u(x(t), t) = u_t + uu_x = 0$$

so that  $u(x(t), t) = u(x_0, 0) = f(x_0)$ . Thus along characteristics

$$(d/dt)x(t, x_0) = u(x(t), t) = f(x_0)$$

and the characteristics satisfy

$$x(t, x_0) = x_0 + f(x_0)t \quad \text{for all } x_0, t > 0 \tag{3}$$

**Step 2:** As in Step 1,  $u(x(t, x_0), t) = f(x_0)$  is constant along characteristics.

If  $x_1 < x_2$  and  $f(x_2) < f(x_1)$ , then by (3) there exists  $t > 0$  such that

$$x_1 + f(x_1)t = x_2 + f(x_2)t \tag{4}$$

For that value of  $t$

$$\begin{aligned} f(x_1) &= u(x(t, x_1), t) = u(x_1 + f(x_1)t, t) \\ &= u(x_2 + f(x_2)t, t) = u(x(t, x_2), t) = f(x_2) \neq f(x_1) \end{aligned}$$

This is a contradiction, which implies that  $u_t + uu_x = 0$  cannot have a continuously-differentiable solution for  $t > 0$  with  $u(x, 0) = f(x)$ . Any function  $u(x, t)$  that is continuously differentiable in both arguments satisfies  $(d/dt)(u(x(t, x_0), t)) = 0$  in this case, which is what leads to the contradiction.

**6.** (Show that  $u_t + uu_x = 0$ ,  $u(x, 0) = f(x)$  has a continuously-differentiable solution for all  $t > 0$  if  $f'(x) > 0$  for all  $x$ .) If  $f'(x) > 0$  for all  $x$ , then  $x_1 < x_2$  implies  $f(x_1) < f(x_2)$ . In particular, equation (4) above cannot happen for  $t > 0$ .

Fix  $t > 0$ . Since  $f'(x) > 0$ , the function  $\phi(x_0) = x_0 + f(x_0)t$  is strictly increasing as a function of  $x_0$ . Since  $\phi(x_0) \rightarrow -\infty$  as  $x_0 \rightarrow -\infty$  and  $\phi(x_0) \rightarrow \infty$  as  $x_0 \rightarrow \infty$ , it follows that the equation

$$x = \phi(x_0) = x_0 + f(x_0)t$$

is uniquely solvable for  $x_0$  for any  $x$  and fixed  $t > 0$ . Writing the unique solution as  $x_0 = p(x, t)$ , we have

$$x = x(t, x_0) \quad \text{if and only if} \quad x_0 = p(x, t) \tag{5}$$

It follows from the fact that

$$\frac{\partial}{\partial x_0}(x_0 + f(x_0)t) = 1 + f'(x_0)t \neq 0$$

for all  $x_0$  and  $t > 0$  that  $p(x, t)$  is continuously differentiable in  $x$  and  $t$ . It then follows from Step 2 of Problem 5 that

$$u(x, t) = f(p(x, t))$$

is a continuously differentiable solution of  $u_t + uu_x = 0$  for all  $t > 0$  with  $u(x, 0) = f(x)$ .