Math450 HW1 Answers — Spring 2009

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1. (a) Exercise 1 and (b) Exercise 3 at the end of Section 2.1, page 21 on the textbook. (Verify that two formulas were solutions of particular partial differential equations. The verifications are not difficult.)

2. (Solve $u_t + cu_x = xt$, $u(x,0) = f(x)$) (An inhomogeneous first-order partial differential equation.) Solving first-order PDEs, both inhomogeneous or homogeneous, can be broken into three steps:

Step 1: Find the characteristics $x(t) = x(t,\ x_0)$ in terms of t and the initial point $x(0) = x_0$. In this case, the characteristic equation is $(d/dt)x = c$ with $x(0) = x_0$, so that the characteristics are

$$x(t) = x(t, x_0) = x_0 + ct \tag{1}$$

The usefulness of characteristics is that

$$(d/dt)u\big(x(t), t\big) = (dx/dt)u_x + u_t = u_t + cu_x = x(t)t$$

where the right-hand side that does not depend on any derivatives, so that finding $v(t) = u\big(x(t), t\big)$ along characteristics reduces to, at worst, solving a second first-order ordinary differential equation.

Step 2: Find $u(x,t)$ along each characteristic. Here

$$(d/dt)u\big(x(t), t\big) = x(t)t = (x_0 + ct)t = x_0 t + ct^2$$

or, if $v(t) = u\big(x(t), t\big)$, then $(d/dt)v(t) = x_0 t + ct^2$. Thus

$$v(t) = u(x(t, x_0),\ t) = (1/2)x_0 t^2 + (1/3)ct^3 + v(0)$$

Here $v(0) = u\big(x(0,\ x_0), 0\big) = u(x_0, 0) = f(x_0)$, so that

$$u\big(x(t),\ t\big) = u(x_0 + ct, t) = (1/2)x_0 t^2 + (1/3)ct^3 + f(x_0)$$

Step 3: Change coordinates from (x_0, t) back to (x,t) to derive a formula for $u(x,t)$ (without reference to characteristics). This can be done by finding the inverse $p(x,t)$ of $x(t,\ x_0)$ for fixed $t > 0$ so that

$$x = x(t, x_0) \quad \text{if and only if} \quad x_0 = p(x,t)$$

In this case, $x = x(t,\ x_0) = x_0 + ct$ if and only if $x_0 = p(x,t)$ for $p(x,t) = x - ct$, so that the solution is

$$\begin{aligned} u(x,t) &= (1/2)(x - ct)t^2 + (1/3)ct^3 + f(x - ct) \\ &= (1/2)xt^2 - (1/6)ct^3 + f(x - ct) \end{aligned}$$

3. (Solve $u_t + \frac{1}{x}u_x = 0$, $u(x,0) = f(x)$, for $x > 0$) Following the three steps of Problem 2,

Step 1: Here the characteristics satisfy $(d/dt)x = 1/x$, or $x(d/dt)x = (1/2)(d/dt)x^2 = 1$, so that $x(t, x_0)^2 = 2t + 2C = 2t + x_0^2$, so that the characteristic with $x(0) = x_0$ is

$$x(t,\ x_0) = \sqrt{x_0^2 + 2t}$$

Step 2: Along characteristics, $(d/dt)u\big(x(t), t\big) = 0$, so that

$$u\big(x(t,\ x_0), t\big) = u(x_0, 0) = f(x_0)$$

Step 3: Change coordinates from (x_0, t) back to (x, t) to derive a formula for $u(x, t)$ without reference to characteristics. If

$$x = x(t, x_0) \qquad \text{if and only if} \qquad x_0 = p(x, t) \tag{2}$$

then $u(x, t) = f\big(p(x, t)\big)$ in this case (and for many homogeneous first-order PDEs in general). Here $x = \sqrt{x_0^2 + 2t}$ so that $x^2 = x_0^2 + 2t$ and $x_0 = p(x, t) = \sqrt{x^2 - 2t}$. Thus

$$u(x, t) = f\big(\sqrt{x^2 - 2t}\,\big)$$

for $x > 0$ and $0 \le t < (1/2)x^2$.

4. (Solve $u_t + cu_x = u^2$, $u(x, 0) = f(x)$) Following the same three steps,
Step 1: As in Problem 2, the characteristics are $x(t, x_0) = x_0 + ct$.
Step 2: $v(t) = u\big(x(t), t\big)$ satisfies

$$(d/dt)v(t) = u_t + cu_x = u\big(x(t), t\big)^2 = v(t)^2$$

or $v(t)^{-2}(d/dt)v(t) = -(d/dt)(1/v(t)) = 1$. Thus $-1/v(t) = t + C$ or

$$v(t) = u(x(t,\ x_0), t) = -1/(t + C) = 1/(C_1 - t), \qquad C_1 = -C$$

Thus

$$v(0) = u(x_0, 0) = f(x_0) = 1/C_1 \qquad \text{and so} \qquad C_1 = 1/f(x_0)$$

which leads to

$$u(x(t,\ x_0), t) = 1/(C_1 - t) = \frac{f(x_0)}{1 - tf(x_0)}$$

Step 3: As in Problem 2, $x_0 = p(x, t) = x - ct$ if and only if $x = x(t,\ x_0) = x_0 + ct$, so that the solution is

$$u(x, t) = \frac{f\big(p(x, t)\big)}{1 - tf\big(p(x, t)\big)} = \frac{f(x - ct)}{1 - tf(x - ct)}$$

which is valid as long as $tf(x - ct) < 1$ along $y(t) = x - ct$ (and otherwise blows up).

5. (Show that $u_t + uu_x = 0$, $u(x,0) = f(x)$ cannot have a continuously-differentiable solution for all $t > 0$ if there exist $x_1 < x_2$ such that also $f(x_2) < f(x_1)$.) Following the recipe above,

Step 1: The characteristics satisfy $(d/dt)x(t, x_0) = u\big(x(t), t\big)$. However, along a characteristic,

$$(d/dt)u\big(x(t), t\big) = u_t + uu_x = 0$$

so that $u\big(x(t), t\big) = u(x_0, 0) = f(x_0)$. Thus along characteristics

$$(d/dt)x(t,\ x_0) = u\big(x(t), t\big) = f(x_0)$$

and the characteristics satisfy

$$x(t, x_0) = x_0 + f(x_0)t \qquad \text{for all } x_0, t > 0 \tag{3}$$

Step 2: As in Step 1, $u(x(t, x_0), t) = f(x_0)$ is constant along characteristics.

If $x_1 < x_2$ and $f(x_2) < f(x_1)$, then by (3) there exists $t > 0$ such that

$$x_1 + f(x_1)t = x_2 + f(x_2)t \tag{4}$$

For that value of t

$$\begin{aligned} f(x_1) &= u\big(x(t, x_1),\ t\big) = u\big(x_1 + f(x_1)t,\ t\big) \\ &= u\big(x_2 + f(x_2)t,\ t\big) = u\big(x(t, x_2),\ t\big) = f(x_2) \neq f(x_1) \end{aligned}$$

This is a contradiction, which implies that $u_t + uu_x = 0$ cannot have a continuously-differentiable solution for $t > 0$ with $u(x, 0) = f(x)$. Any function $u(x, t)$ that is continuously differentiable in both arguments satisfies $(d/dt)\big(u(x(t, x_0)\, t\big) = 0$ in this case, which is what leads to the contradiction.

6. (Show that $u_t + uu_x = 0$, $u(x, 0) = f(x)$ has a continuously-differentiable solution for all $t > 0$ if $f'(x) > 0$ for all x.) If $f'(x) > 0$ for all x, then $x_1 < x_2$ implies $f(x_1) < f(x_2)$. In particular, equation (4) above cannot happen for $t > 0$.

Fix $t > 0$. Since $f'(x) > 0$, the function $\phi(x_0) = x_0 + f(x_0)t$ is strictly increasing as a function of x_0. Since $\phi(x_0) \to -\infty$ as $x_0 \to -\infty$ and $\phi(x_0) \to \infty$ as $x_0 \to \infty$, it follows that the equation

$$x = \phi(x_0) = x_0 + f(x_0)t$$

is uniquely solvable for x_0 for any x and fixed $t > 0$. Writing the unique solution as $x_0 = p(x,t)$, we have

$$x = x(t, x_0) \quad \text{if and only if} \quad x_0 = p(x,t) \tag{5}$$

It follows from the fact that

$$\frac{\partial}{\partial x_0}(x_0 + f(x_0)t) = 1 + f'(x_0)t \neq 0$$

for all x_0 and $t > 0$ that $p(x,t)$ is continuously differentiable in x and t. It then follows from Step 2 of Problem 5 that

$$u(x,t) = f\big(p(x,t)\big)$$

is a continuously differentiable solution of $u_t + uu_x = 0$ for all $t > 0$ with $u(x,0) = f(x)$.