Ma 5051 — Real Variables and Functional Analysis

Problem Set #1 — Due September 10, 2009

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Text references are to Gerald B. Folland, *Real Analysis: Modern Techniques and Their Applications*, 2nd edn, John Wiley & Sons, 1999.

1. (Problem 5, page 24) Let $\mathcal{M} = \mathcal{M}(\mathcal{E})$ be the σ-algebra of subsets of X generated by subsets $\mathcal{E} \subseteq P(X)$. Then

$$\mathcal{M} \;=\; (\text{Set union})\{\, \mathcal{M}(\mathcal{C}_0) : \text{countable } \mathcal{C}_0 \subseteq \mathcal{E}\,\}$$

where $\mathcal{M}(\mathcal{C})$ means the smallest σ-algebra containing $\mathcal{C}$. (*Hint*: Show that the right-hand side above is a σ-algebra.)

2. Let $X = (X, \rho)$ be a separable metric space and set $S_r(x) = \{\, y : \rho(x, y) < r\,\}$ for real $r > 0$ and $x \in X$. Then the Borel σ-algebra $\mathcal{B}(X)$ (generated by the class τ of all open sets $\mathcal{O} \subseteq X$) satisfies

$$\mathcal{M}\{\, S_r(x) : r > 0,\, x \in X\,\} \;=\; \mathcal{B}(X)$$

(Note that this may not be true if X is not separable.)

3. (Problem 12, page 27) Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Recall that $E \vartriangle F = (E - F) \cup (F - E)$.

(a) If $E, F \in \mathcal{M}$ and $\mu(E \vartriangle F) = 0$, then $\mu(E) = \mu(F)$.

(b) Say that $E \sim F$ if $\mu(E \vartriangle F) = 0$. Then $\sim$ is an equivalence relation on $\mathcal{M}$.

(c) Define $\rho(E, F) = \mu(E \vartriangle F)$ if $E, F \in \mathcal{M}$. Then $\rho(E, G) \le \rho(E, F) + \rho(F, G)$ for all $E, F, G \in M$ and hence ρ defines a metric on the space $\mathcal{M}/\!\sim$ of equivalence classes.

4. Set $\mu^*(E) = \sqrt{\mathrm{card}(E)}$ for subsets $E \subseteq X$ where $\mathrm{card}(E)$ means the number of elements of E. (Thus $0 \le \mathrm{card}(E) \le \infty$.)

(a) Show that $\mu^*(E)$ is an outer measure on X.

(b) Define $\mathcal{F}(\mu^*)$ as the class of all μ^*-measurable subsets $A \subseteq X$ (see Section 1.4 in the text). Then $\mathcal{F}(\mu^*) = \{\, \phi, X\,\}$. (*Hint*: Consider sets E with two points.)

Carathéodory's Theorem (Theorem 1.11 in the text) asserts that, for any outer measure μ^*, $\mathcal{F}(\mu^*)$ is a σ-algebra and μ^* is a measure on $\mathcal{F}(\mu^*)$. Note that the conclusion in this case is not deep.