Ma 5051 — Real Variables and Functional Analysis

Problem Set #2 — Due September 17, 2009

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1. (Problem 17, page 32) If μ^* is an outer measure on X and $\{ A_j : j = 1, 2, \ldots \}$ is a sequence of disjoint μ^*-measurable sets, then

$$\mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j \right) \right) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j) \qquad \text{for any set } E \subseteq X$$

2. (Problem 18, page 32) Let $\mathcal{A}$ be an algebra of subsets of X, $\mathcal{A}_\sigma$ the collection of sets formed from countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ the collection of sets formed from countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$ and let μ^* be the outer measure on $P(X)$ induced by μ_0.

(a) For any $E \subseteq X$ and $\epsilon > 0$, there exists $A \in \mathcal{A}_\sigma$ such that $E \subseteq A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$.

(b) If $\mu^*(E) < \infty$, then E is μ^*-measurable iff there exists some $B \in \mathcal{A}_{\sigma\delta}$ such that $E \subseteq B$ and $\mu^*(B - E) = 0$. (*Hint*: Show C is μ^*-measurable if $\mu^*(C) = 0$, and that any subset $C_2 \subseteq C$ satisfies $\mu^*(C_2) = 0$ and so is also μ^*-measurable.)

(c) If X is σ-finite, then the conclusion of (b) holds without the condition that $\mu^*(E) < \infty$.

3. (Problem 19, page 32) Let μ^* be the outer measure on X induced from a premeasure μ_0 on X with $\mu_0(X) < \infty$. Define $\mu_*(E) = \mu_0(X) - \mu^*(E^c)$ for all $E \subseteq X$. Then E is μ^*-measurable iff $\mu^*(E) = \mu_*(E)$. (*Hint*: Use the previous problem.)

4. (Like Problem 24, page 33) Let μ be a measure on $(X, \mathcal{M})$ with $\mu(X) < \infty$. Assume $E \subseteq X$ with $\mu^*(E) = \mu(X)$ and $E \notin \mathcal{M}$, where μ^* is the outer measure induced by μ on $\mathcal{M}$. (In this case, E is called a "non-measurable set of full outer measure".)

(a) Show that $\mu^*(A \cap E) = \mu(A)$ for all $A \in \mathcal{M}$. (*Hint*: Note that $\mu^*(A \cap E) + \mu^*(A^c \cap E) = \mu^*(E)$ by the definition of μ^*-measurability.)

(b) Let $\mathcal{M}_E = \{ A \cap E : A \in \mathcal{M} \}$. (These are the sets in $\mathcal{M}$ restricted to E.) Define $\nu(F) = \mu^*(F)$ for $F \in \mathcal{M}_E$. Show that $\mathcal{M}_E$ is a σ-algebra on E and that ν is a measure on $\mathcal{M}_E$.

Remark: This shows that you can throw away all of the points in X that are not in a "set of full outer measure" in X and retain essentially the same measure space. In some cases, E is a set of smaller cardinality than X.