Ma 5051 — Real Variables and Functional Analysis

Solutions for Problem Set #3 due September 24, 2009

Prof. Sawyer — Washington University

See HOMEWORK#3 on the Math 5051 Web site for the text of the problems.

1. Recall $0 < m(E) < \infty$ and assume $0 < \alpha < 1$. If no such open interval I exists, then $m(E \cap I) \le \alpha m(I)$ for all open intervals $I = (a, b)$. Since $m(\{ x \}) = 0$ for Lebesgue measure, the same holds for cells $I = (a, b]$. Choose $\epsilon > 0$ such that $\alpha(1 + \epsilon) < 1$ and then choose disjoint cells $I_j = (a_j, b_j]$ such that $E \subseteq B = \bigcup_{j=1}^{\infty} I_j$ and $\sum_{j=1}^{\infty} m(I_j) = m(B) < m(E)+m(E)\epsilon = m(E)(1+\epsilon)$. Then $m(E\cap I_j) \le \alpha m(I_j)$ for all j and $\sum_{j=1}^{\infty} m(E \cap I_j) = m(E \cap B) = m(E) \le \alpha \sum_{j=1}^{\infty} m(I_j) = \alpha m(B) \le \alpha(1 + \epsilon)m(E) < m(E)$, which contradicts $m(E) > 0$.

2. Choose disjoint cells $I_j = (a_j, b_j]$ such that $E \subseteq B = \bigcup_{j=1}^{\infty} I_j$ and $\mu(B) < \mu(E) + \epsilon/2$. Then $\mu(B - E) = \mu(B) - \mu(E) < \epsilon/2$. Let $B_n = \bigcup_{j=1}^{n} I_j$ where $\sum_{j=n+1}^{\infty} \mu(I_j) < \epsilon/2$. Then $\mu(B_n - E) \le \mu(B - E) < \epsilon/2$ and $E - B_n \subseteq B - B_n = \bigcup_{j=n+1}^{\infty} I_j$ so that $\mu(E - B_n) < \epsilon/2$. Since $E \bigtriangleup B_n = (E - B_n) \cup (B_n - E)$, we have $\mu(E \bigtriangleup B_n) = \mu(E - B_n) + \mu(B_n - E) < \epsilon$.

3. Since $a_i \le b_i$, the differences $c_i = b_i - a_i \ge 0$. By assumption, $\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} (b_i - a_i) = \sum_{i=1}^{n} b_i - \sum_{i=1}^{n} a_i = 0$. If any $c_i > 0$, then $\sum_{j=1}^{n} c_j \ge c_i > 0$, which would be a contradiction, so that $c_i = 0$ (and $a_i = b_i$) for $1 \le i \le n$.

4. (i) $F(x)$ has jumps of size 1 ($F(x+)-F(x-) = 1$) at all integers and is otherwise continuous. Thus $\mu_F(\{n\}) = 1$ and $\mu_F(\{a\}) = 0$ at all other values $a \in R$.

(ii) Since $F(n-) - F(n - 1) = 0$ for all n, $\mu_F(\,(n, n + 1)\,) = 0$ for all open intervals $(n, n + 1)$. By (i), $\mu_F(\{n\}) = 1$ for all integers n. The set A contains intervals around the points $1/2, 1, 3/2, 2, 5/2$ and, in particular, contains the points $1, 2$ along with subsets of the open intervals $(0, 1)$, $(1, 2)$, and $(2, 3)$ (which are μ_F-null sets). Thus $\mu_F(A) = \mu_F(\{1\}) + \mu_F(\{2\}) = 2$.

5. (i) Note (a) $\phi \in \Gamma_Q$, (b) $C_1, C_2 \in \Gamma_Q$ implies $C_1 \cap C_2 \in \Gamma_Q$, and (c) $C_1, C_2 \in \Gamma_Q$ implies $C_1 - C_2 = \bigcup_{j=1}^{m} D_j$ for disjoint $D_j \in \Gamma_Q$ and $m = 0$, 1, or 2 by the same arguments as in the case of cells for dimension $k = 1$. The set function $\nu(A)$ is finitely additive on Γ_Q for the same reason.

(ii) For $x \in Q$, $\nu^*(\{x\}) = \inf\{ \sum_{i=1}^{\infty} \nu(\,(a_i, b_i]_Q\,) : x \in \bigcup_{j=1}^{\infty} (a_i, b_i]_Q \}$. In particular $x \in (x - 1/n, x + 1/n]_Q$ for $n \ge 1$ implies

$$\nu^*(\{x\}) \;\le\; \nu(\,(x - 1/n, x + 1/n]_Q\,) \;=\; 2/n$$

for all $n \geq 1$, which implies $\nu^*(\{x\}) = 0$.

(iii) Every $E \subseteq Q$ is countable, so that, if $E = \{ q_n \}$ for $q_n \in E$, then $\nu^*(E) \leq \sum_{n=1}^{\infty} \nu^*(\{q_n\}) = 0$.

(iv) If ν were a premeasure on Γ_Q, then Proposition 1.13 would imply that $\nu^*((a, b]_Q) = \nu((a, b]_Q) = b - a$, which is false by (iii) if $a < b$. Thus ν cannot be a premeasure. The proof of Proposition 1.15 that a set function like ν is a premeasure requires that closed and bounded intervals be compact, which is not true in Q. Thus the proof of Proposition 1.15 breaks down at the step requiring compactness. Further steps that assume this are generally false, but all other steps in the proofs seem OK.