Ma 5051 — Real Variables and Functional Analysis

Problem Set #4 — Due October 1, 2009

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In the following, assume that $(X, \mathcal{M})$ is a measurable space: That is, X is a set and $\mathcal{M}$ a σ-algebra of subsets of X. Define $\int_E f(x)d\mu = \int I_E(x)f(x)d\mu$ for measures μ on $\mathcal{M}$, $E \in \mathcal{M}$, and $f \in L^+(X, \mathcal{M})$.

1. (Problem 1, page 48) Assume $f : X \to \overline{R} = [-\infty, \infty]$. Prove that $f(x)$ is $(\mathcal{M}, \mathcal{B}(\overline{R}))$-measurable if and only if $f^{-1}(\{-\infty\}) = \{x : f(x) = -\infty\} \in \mathcal{M}$, $f^{-1}(\{\infty\}) \in \mathcal{M}$, and $f(x)$ is $(\mathcal{M}, \mathcal{B}(R))$-measurable on $Y = \{x : |f(x)| < \infty\}$.

2. (Problem 3, page 48) If $\{f_n\}$ is a sequence of $\mathcal{M}$-measurable functions on X, then $Y = \{x : \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is a measurable set.

3. (Problem 8, page 48) If $f : R \to R$ is monotone, then $f(x)$ is Borel measurable. (*Hint*: Be careful!)

4. (Problem 14, page 52) For $f \in L^+(X, \mathcal{M})$, define $\nu(E) = \int_E f(x)d\mu$ for $E \in \mathcal{M}$. Prove that

(a) ν is a measure on $\mathcal{M}$

(b) For any $g \in L^+(X, \mathcal{M})$, $\int g(x)d\nu = \int g(x)f(x)d\mu$. (*Hint*: First suppose that g is simple. Proposition 2.10, which we'll cover next Tuesday, might be helpful.)

5. (Problem 16, page 52) Assume $f \in L^+(X, \mathcal{M})$ with $\int f(x)d\mu < \infty$ and perhaps $\mu(X) = \infty$. Prove that, for every $\epsilon > 0$, there exists some $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_E f(x)d\mu > \int f(x)d\mu - \epsilon$. (Do not assume that X is σ-finite.)