In the following, assume that \((X, \mathcal{M})\) is a measurable space: That is, \(X\) is a set and \(\mathcal{M}\) a \(\sigma\)-algebra of subsets of \(X\). Define \(\int_E f(x) d\mu = \int I_E(x) f(x) d\mu\) for measures \(\mu\) on \(\mathcal{M}\), \(E \in \mathcal{M}\), and \(f \in L^+(X, \mathcal{M})\).

1. (Problem 1, page 48) Assume \(f : X \to \mathbb{R} = [-\infty, \infty]\). Prove that \(f(x)\) is \((\mathcal{M}, \mathcal{B}(\mathbb{R}))\)-measurable if and only if \(f^{-1}(-\infty) = \{ x : f(x) = -\infty \} \in \mathcal{M}\), \(f^{-1}(\{ \infty \}) \in \mathcal{M}\), and \(f(x)\) is \((\mathcal{M}, \mathcal{B}(\mathbb{R}))\)-measurable on \(Y = \{ x : |f(x)| < \infty \}\).

2. (Problem 3, page 48) If \(\{ f_n \}\) is a sequence of \(\mathcal{M}\)-measurable functions on \(X\), then \(Y = \{ x : \lim_{n \to \infty} f_n(x) \text{ exists} \}\) is a measurable set.

3. (Problem 8, page 48) If \(f : \mathbb{R} \to \mathbb{R}\) is monotone, then \(f(x)\) is Borel measurable. (Hint: Be careful!)

4. (Problem 14, page 52) For \(f \in L^+(X, \mathcal{M})\), define \(\nu(E) = \int_E f(x) d\mu\) for \(E \in \mathcal{M}\). Prove that
   (a) \(\nu\) is a measure on \(\mathcal{M}\)
   (b) For any \(g \in L^+(X, \mathcal{M})\), \(\int g(x) d\nu = \int g(x) f(x) d\mu\). (Hint: First suppose that \(g\) is simple. Proposition 2.10, which we’ll cover next Tuesday, might be helpful.)

5. (Problem 16, page 52) Assume \(f \in L^+(X, \mathcal{M})\) with \(\int f(x) d\mu < \infty\) and perhaps \(\mu(X) = \infty\). Prove that, for every \(\epsilon > 0\), there exists some \(E \in \mathcal{M}\) such that \(\mu(E) < \infty\) and \(\int_E f(x) d\mu > \int f(x) d\mu - \epsilon\). (Do not assume that \(X\) is \(\sigma\)-finite.)