Ma 5051 — Real Variables and Functional Analysis

Model Solutions for Problem Set #4 due October 1, 2009

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In the following, assume that $(X, \mathcal{M})$ is a measurable space: That is, X is a set and $\mathcal{M}$ a σ-algebra of subsets of X. Define $\int_E f(x)d\mu = \int I_E(x)f(x)d\mu$ for measures μ on $\mathcal{M}$, $E \in \mathcal{M}$, and $f \in L^+(X, \mathcal{M})$.

1. According to page 43 of the text, a mapping $f : X \to \overline{R} = [-\infty, \infty]$ for a measurable space $(X, \mathcal{M})$ is *measurable* (or $(\mathcal{M}, \mathcal{B}(\overline{R}))$*-measurable*) if $f^{-1}(A) = \{ x : f(x) \in A \} \in \mathcal{M}$ for all sets $A \in \mathcal{B}(\overline{R})$. Here $\mathcal{B}(\overline{R})$ is the Borel sets in $\overline{R}$, which is the same as the Borel hull of (that is, the smallest σ-algebra containing) $\mathcal{E} = \{ [-\infty, x] : x \in R \}$. Note that $\{ -\infty \} = \bigcap_{n=1}^{\infty} [-\infty, -n]$ and $\{ \infty \} = (\bigcup_{n=1}^{\infty} [-\infty, n])^c$, and that $\mathcal{B}(\overline{R})$ is the smallest σ-algebra containing $\mathcal{B}(R)$ and the two sets $\{ \pm\infty \}$ containing one point each.

(a) If $f(x)$ is $(\mathcal{M}, \mathcal{B}(\overline{R}))$-measurable, then $A = A_- = \{ -\infty \}$, $A = A_+ = \{ \infty \}$, and all sets $A \in \mathcal{B}(R)$ satisfy $A \in \mathcal{B}(\overline{R})$. Thus $f^{-1}(A) \in \mathcal{M}$ in all three cases.

(b) If a collection of sets $\mathcal{E} \subseteq \mathcal{B}(\overline{R})$ satisfies (i) $f^{-1}(A) \in \mathcal{M}$ for all $A \in \mathcal{E}$ and (ii) $\mathcal{B}(\mathcal{E})$ (the Borel hull of $\mathcal{E}$) equals $\mathcal{B}(\overline{R})$, as is the case here, then f is $(\mathcal{M}, \mathcal{B}(\overline{R}))$-measurable.

2. Proof I (of two different proofs): The point $x \in Y$ if and only if $f_n(x)$ converges as $n \to \infty$, which is the same as saying that $\{ f_n(x) \}$ satisfies Cauchy's condition. Thus

$$Y = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{a \geq m} \bigcap_{b \geq m} \{ x : |f_a(x) - f_b(x)| > 1/k \}$$

This is measurable since each set $\{ x : |f_a(x) - f_b(x)| > 1/k \}$ is measurable.

Proof II: By Proposition (2.7), both $g_1(x) = \liminf_{n\to\infty} f_n(x)$ and $g_2(x) = \limsup_{n\to\infty} f_n(x)$ are measurable. Thus $Y = \{ x : g_1(x) = g_2(x) \} = \bigcap_{k=1}^{\infty} \{ x : g_2(x) - g_1(x) < 1/k \}$ is also measurable.

3. Assume for definiteness that $f(x)$ is monotonically increasing.
Let $E_\lambda = f^{-1}((-\infty, \lambda]) = \{ x : f(x) \leq \lambda \}$. Then $x_1 < x_2$ and $x_2 \in E_\lambda$ implies $f(x_1) \leq f(x_2) \leq \lambda$ and hence $x_1 \in E_\lambda$. If $x_{\max} = \sup\{ x : x \in E_\alpha \}$, then either $E_\lambda = (-\infty, x_{\max}) \in \mathcal{B}(R)$ or $E_\lambda = (-\infty, x_{\max}] \in \mathcal{B}(R)$ (both can occur). Hence $f(x)$ is Borel measurable.

4. (a) The set function $\nu(E) = \int_E f(x)d\mu$ for $E \in \mathcal{M}$ satisfies (i) $\nu(E) \geq 0$ for all $E \in \mathcal{M}$ since $f \in \mathcal{L}^+(X, \mathcal{M})$ and (ii) $\nu(\phi) = 0$. Now assume $E = \bigcup_{n=1}^{\infty} E_n$ where

$E_n \in \mathcal{M}$ are disjoint. Then $I_E(x) = \sum_{n=1}^{\infty} I_{E_n}(x)$ and

$$\begin{aligned} \nu(E) \ &= \ \int I_E(x)f(x)d\mu \ = \ \int \sum_{n=1}^{\infty} I_{E_n}(x)f(x)d\mu \\ &= \ \sum_{n=1}^{\infty} \int I_{E_n}(x)f(x)d\mu \ = \ \sum_{n=1}^{\infty} \nu(E_n) \end{aligned}$$

by Theorem 2.15, which is equivalent to the monotone convergence theorem. Thus $\nu(E)$ is a measure on $\mathcal{M}$.

(b) Let $\phi(x) = \sum_{j=1}^{M} c_j I_{E_j}(x)$ be a simple functions with $c_j \geq 0$. Since $\nu(E_j) = \int_{E_j} f(x)d\mu = \int I_{E_j}(x)f(x)\,d\mu$, it follows that $\int \phi(x)d\nu = \sum_{j=1}^{M} c_j \nu(E_j) = \sum_{j=1}^{M} c_j \int I_{E_j}(x)f(x)d\mu = \int \phi(x)f(x)d\mu$. If $g \in L^{+}(X, \mathcal{M})$ is arbitrary, there exist simple functions with $0 \leq \phi_n(x) \uparrow g(x)$ for all x by Theorem 2.10. It then follows from the monotone convergence theorem that $\int g(x)d\nu = \int g(x)f(x)d\mu$.

5. Let $f_n(x) = f(x)I_{E_n}(x)$ for $E_n = \{\, f(x) > 1/n \,\}$. Then

$$\mu(E_n) \ \leq \ n \int f(x)d\mu) \ < \ \infty$$

and $0 \leq f_n(x) \uparrow f(x)$ as $n \to \infty$. Thus, by the monotone convergence theorem, $0 \leq \int f_n(x)d\mu = \int_{E_n} f(x)d\mu \uparrow \int f(x)d\mu < \infty$. Since $\int f(x)d\mu < \infty$, we can choose n such that $\int f(x)d\mu - \int_{E_n} f(x)d\mu < \epsilon$ and set $E = E_n$.