Ma 5051 — Real Variables and Functional Analysis

Problem Set #5 — Due October 8, 2009

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Let $(X, \mathcal{M}, \mu)$ be a measure space. Recall $\int_A f(x)d\mu = \int I_A(x)f(x)d\mu$ for $A \in \mathcal{M}$ and $f \in L^+ \cup L^1$, where $I_A(x)$ is the indicator function of A.

1. (Problem 20, page 59) Assume $f_n, g_n, f, g \in L^1$, $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ a.e. as $n \to \infty$, $|f_n(x)| \le g_n(x)$, and $\int g_n(x)d\mu \to \int g(x)d\mu$. Then show $\int f_n(x)d\mu \to \int f(x)d\mu$. (*Hint*: Rework the proof of the dominated convergence theorem.

2. (Problem 21, page 59) Assume $f_n, f \in L^1$ and $f_n(x) \to f(x)$ a.e. Then show that $\int |f_n - f|d\mu \to 0$ if and only if $\int |f_n(x)|d\mu \to \int |f(x)|d\mu$. (*Hint*: Use the previous problem.)

3. (Problem 26, page 60) If $f \in L^1\big(R, \mathcal{B}(R), m\big)$ for Lebesgue measure m and $F(x) = \int_{-\infty}^x f(y)dy$, then $F(x)$ is a continuous function on R.

4. (Problem 28ac, page 60) Compute the following limits and justify the calculations:

(a) $\lim_{n\to\infty} \int_0^\infty \big(1 + (x/n)\big)^{-n} \sin(x/n)dx$

(c) $\lim_{n\to\infty} \int_0^\infty n\sin(x/n)[x(1+x^2)]^{-1}dx$

5. (Problem 31ac, page 60) Derive the following formulas by expanding part of the integrand into an infinite series and justifying term-by-term integration. Exercise 29 may be useful.

(a) For $a > 0$, $\int_{-\infty}^\infty e^{-x^2}\cos(ax)dx = \sqrt{\pi}\, e^{-a^2/4}$

(c) For $a > 1$, $\int_0^\infty x^{a-1}(e^x - 1)^{-1}dx = \Gamma(a)\zeta(a)$, where $\zeta(a) = \sum_{n=1}^\infty n^{-a}$.