Let \((X, \mathcal{M}, \mu)\) be a measure space. Recall \(\int_A f(x) \, d\mu = \int I_A(x) f(x) \, d\mu\) for \(A \in \mathcal{M}\) and \(f \in L^+ \cup L^1\), where \(I_A(x)\) is the indicator function of \(A\).

1. Since \(g_n(x) \pm f_n(x) \geq 0\), \(g_n(x) \rightarrow g(x)\) a.e., and \(f_n(x) \rightarrow f(x)\) a.e., by Fatou’s Lemma
   \[
   \int (g(x) + f(x)) \, dx \leq \liminf_{n \to \infty} \int (g_n(x) + f_n(x)) \, dx
   \]
   \[
   \int (g(x) - f(x)) \, dx \leq \liminf_{n \to \infty} \int (g_n(x) - f_n(x)) \, dx
   \]
   Subtract the limit \(\int g_n(x) \, d\mu \rightarrow \int g(x) \, d\mu\) from both inequalities and note that the second inequality is equivalent to \(\limsup_{n \to \infty} \int f_n(x) \, d\mu \leq \int f(x) \, d\mu\). Then
   \[
   \int f(x) \, d\mu \leq \liminf_{n \to \infty} \int f_n(x) \, d\mu \leq \limsup_{n \to \infty} \int f_n(x) \, d\mu \leq \int f(x) \, d\mu
   \]
   Since the first and last integrals are the same and finite, \(\int f_n(x) \, d\mu \rightarrow \int f(x) \, d\mu\).

2. (a) If \(\int |f_n - f| \, d\mu \rightarrow 0\), then
   \[
   \left| \int |f_n| \, d\mu - \int |f| \, d\mu \right| = \left| \int (|f_n| - |f|) \, d\mu \right| \leq \int |f_n| - |f| \, d\mu \leq \int |f_n - f| \, d\mu \rightarrow 0
   \]
   (b) Let \(g_n(x) = |f_n(x)| + |f(x)|\) and \(g(x) = 2|f(x)|\). Then \(|f_n(x) - f(x)| \leq g_n(x)\), \(|f_n(x) - f(x)| \rightarrow 0\) a.e., \(g_n(x) \rightarrow g(x)\) a.e., and \(\int g_n \, d\mu \rightarrow \int g \, d\mu\) since \(\int |f_n| \, d\mu \rightarrow \int |f| \, d\mu\). Hence, by the previous problem, \(\int |f_n - f| \, d\mu \rightarrow 0\).

3. It is sufficient to show that \(x_n \rightarrow x\) implies \(F(x_n) \rightarrow F(x)\) where
   \[
   F(x) = \int_{-\infty}^x f(y) \, dy = \int I_{(-\infty, x)}(y) f(y) \, dy
   \]
   Note that \(I_{(0, x_n)}(y) f(y) \rightarrow I_{(0, x)}(y) f(y)\) for \(y \neq x\) (that is, for a.e. \(y\) and
   \[
   |I_{(-\infty, x)}(y) f(y)| \leq |f(y)|
   \]
   since \(f \in L^1\) implies \(\int |f(y)| \, dy < \infty\). Thus \(F(x_n) \rightarrow F(x)\) by the dominated convergence theorem.
4. (Problems 28ac, page 60 in text)

(a) Evaluate \( \lim_{n \to \infty} \int_0^\infty (1 + (x/n))^{-n} \sin(x/n) \, dx \). Proofs I and II below are from student homeworks, in comparison with my more long-winded Proof III. There is also a shorter Proof IV.

In all cases, let \( f_n(x) = (1 + (x/n))^{-n} \sin(x/n) \) and \( g_n(x) = (1 + (x/n))^{-n} \). Then \( |f_n(x)| \leq g_n(x) \), \( f_n(x) \to e^{-x} = 0 \) and \( g_n(x) \to g(x) = e^{-x} \) for all \( x > 0 \), but \( g_n(x) \) is not uniformly bounded by \( e^{-x} \) since \( g_n(x) \sim C_n/x^n \gg e^{-x} \) for fixed \( n \) as \( x \to \infty \).

In Proofs I–III, the problem is to find dominating function(s) for \( g_n(x) \). Proof IV uses a direct estimate of \( |f_n(x)| \) and doesn’t really use the dominated convergence theorem.

If we can show (i) \( g_n(x) \leq g(x) \) where \( \int_0^\infty g(x) \, dx < \infty \), then \( \int_0^\infty f_n \, dx \to 0 \) by dominated convergence. If (ii) \( \int_0^\infty g_n(x) \, dx \to \int_0^\infty g(x) \, dx < \infty \) where \( g_n(x) \to g(x) = e^{-x} \), then we can use Problem 1 to show \( \int_0^\infty f_n(x) \, dx \to \int_0^\infty 0 \, dx = 0 \).

**Proof I:** By the binomial theorem for \( x > 0, n \geq 4 \), \( (1 + (x/n))^n = 1 + n(x/n) + (n/2)(x/n)^2 + \ldots \geq 1 + ((n-1)/2n)x^2 \) or \( \leq 1 + (n(n-1)(n-2)(n-3)/24)(x/n)^4 \). Thus \( (1 + (x/n))^n \geq 1 + (1/4)x^2 \) or \( \leq 1 + (1/256)x^4 \) for \( n \geq 4 \). Thus \( |(1 + (x/n))^{-n} \sin(x/n)| \leq 4/(4 + x^2) \) or \( \leq 256/(256 + x^4) \) and we can use dominated convergence.

**Proof II:** We use the “moving dominated convergence” result of Problem 1 HW5. Let \( f_n(x) \) and \( g_n(x) \) be as above. Then \( |f_n(x)| \leq g_n(x) \), \( g_n(x) \to g(x) = e^{-x} \), and \( f_n(x) \to e^{-x}(0) = 0 \) for all \( x \). If we can then show that \( \lim_{n \to \infty} \int_0^\infty g_n(x) \, dx = \int_0^\infty g(x) \, dx = \int_0^\infty e^{-x} \, dx = 1 \), then we can conclude that \( \int_0^\infty f_n(x) \, dx \to 0 \).

Now \( \int_0^\infty (1 + (x/n))^{-n} \, dx = n \int_0^\infty (1 + x)^{-n} \, dx = n \int_1^\infty x^{-n} \, dx = n/(n-1) \to 1 \) as \( n \to \infty \). Since \( |f_n(x)| \leq g_n(x) \), we conclude \( \int_0^\infty f_n(x) \, dx \to 0 \).

**Proof III:** The following argument is longer in this case, but is easier in other cases. I claim that, for fixed \( x > 0 \), \( (1 + (x/n))^{-n} \downarrow \) as \( n \uparrow \). This would imply (for example) that \( e^{-x} \leq (1 + (x/n))^{-n} \leq (1 + (x/2))^{-2} \) for fixed \( x > 0 \) and \( n \geq 2 \). The statement is equivalent to \( K(y) = (1 + xy)^{-1/y} \) as \( y \uparrow \). Note

\[
\frac{d}{dy} \log K(y) = -\frac{d}{dy} \log(1 + xy) = \frac{\log(1 + xy) - yx/(1 + xy)}{y^2}
\]

For \( A > 0 \), \( \log(1 + A) = \int_1^{1+A} (1/y) \, dy > A/(1 + A) \). Thus, for \( A = xy \), we conclude \( (d/dy) \log K(y) > 0 \) for \( x > 0, y > 0 \). It follows that \( (1 + (x/n))^{-n} \leq (1 + x/2)^{-2} \) for \( n \geq 2 \) and \( x > 0 \). Thus for \( n \geq 2 \)

\[
\left| \frac{\sin(x/n)}{(1 + (x/n))^n} \right| \leq \frac{1}{(1 + x/2)^2}
\]
which is integrable on \((0, \infty)\). Hence by dominated convergence
\[
\lim_{n \to \infty} \int_{0}^{\infty} \frac{\sin(x/n)}{(1 + (x/n))^n} \, dx = \int_{0}^{\infty} \lim_{n \to \infty} \frac{\sin(x/n)}{(1 + (x/n))^n} \, dx = 0
\]

**Proof IV:** More directly, if \(n \geq 3\), the integral is bounded by
\[
\int |(1 + (x/n))^{-n} \sin(x/n)| \, dx \leq \int_{0}^{\infty} (1 + (x/n))^{-n} (x/n) \, dx
\]
\[
= n \int_{0}^{\infty} (1 + x)^{-n} x \, dx = n \int_{1}^{\infty} x^{-n} (x - 1) \, dx
\]
\[
= n \int_{1}^{\infty} (x^{-n+1} - x^{-n}) \, dx = n \left( \int_{1}^{\infty} x^{-n+1} \, dx - \int_{1}^{\infty} x^{-n} \, dx \right)
\]
\[
= n \left( \frac{1}{n - 2} - \frac{1}{n - 1} \right) = \frac{n}{(n - 1)(n - 2)} \to 0
\]

(c) This integral is \(\int_{0}^{\infty} \frac{\sin(x/n)}{x/n} \frac{dx}{1 + x^2} \) where, since \(|\sin(x)/x| \leq 1\),
\[
\left| \frac{\sin(x/n)}{x/n} \frac{1}{1 + x^2} \right| \leq \frac{1}{1 + x^2}
\]
which is integrable on \((0, \infty)\). Thus by dominated convergence
\[
\lim_{n \to \infty} \int_{0}^{\infty} \frac{\sin(x/n)}{x/n} \frac{dx}{1 + x^2} = \int_{0}^{\infty} \lim_{n \to \infty} \frac{\sin(x/n)}{x/n} \frac{dx}{1 + x^2} = \int_{0}^{\infty} \frac{dx}{1 + x^2} = \frac{\pi}{2}
\]

5. (Problem 31ac page 60)

(a) For \(a > 0\),
\[
\int_{-\infty}^{\infty} e^{-x^2} \cos(ax) \, dx = \int_{-\infty}^{\infty} e^{-x^2} \sum_{n=0}^{\infty} \frac{(-1)^n (ax)^{2n}}{(2n)!} \, dx = \int_{-\infty}^{\infty} e^{-x^2} \sum_{n=0}^{\infty} f_n(x) \, dx
\]
where \(f_n(x) = (-1)^n (ax)^{2n}/(2n!)\). The problem is to justify interchanging the integral and the infinite sum. Note
\[
\int_{-\infty}^{\infty} e^{-x^2} \sum_{n=0}^{\infty} |f_n(x)| \, dx = \int_{-\infty}^{\infty} e^{-x^2} \sum_{n=0}^{\infty} \frac{(ax)^{2n}}{(2n)!} \, dx
\]
\[
= \int_{-\infty}^{\infty} e^{-x^2} \frac{(e^{ax} + e^{-ax})}{2} \, dx < \infty
\]
Thus, by a form of dominated convergence (see Theorem 2.25 in the text), we can interchange the sum and the integral and obtain

\[
\sum_{n=0}^{\infty} (-1)^n \frac{a^{2n}}{(2n)!} \int_{-\infty}^{\infty} e^{-x^2} x^{2n} \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{a^{2n}}{(2n)!} \frac{(2n)! \sqrt{\pi}}{4^n n!} = \sqrt{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \left( \frac{a^2}{4} \right)^n = \sqrt{\pi} e^{-a^2/4}
\]

(c) For \( a > 1 \), since the terms in the sum below are nonnegative, and by a form of the monotone convergence theorem (see Theorem 2.15 in the text)

\[
\int_{0}^{\infty} x^{a-1} \frac{1}{e^x - 1} \, dx = \int_{0}^{\infty} x^{a-1} \frac{e^{-x}}{1 - e^{-x}} \, dx = \int_{0}^{\infty} x^{a-1} \sum_{n=1}^{\infty} e^{-nx} \, dx = \sum_{n=1}^{\infty} \int_{0}^{\infty} x^{a-1} e^{-nx} \, dx = \frac{1}{n^a} \int_{0}^{\infty} x^{a-1} e^{-x} \, dx = \Gamma(a) \zeta(a)
\]