Ma 5051 — Real Variables and Functional Analysis

Solutions for Problem Set #5 due October 8, 2009

Let $(X, \mathcal{M}, \mu)$ be a measure space. Recall $\int_A f(x)d\mu = \int I_A(x)f(x)d\mu$ for $A \in \mathcal{M}$ and $f \in L^+ \cup L^1$, where $I_A(x)$ is the indicator function of A.

1. Since $g_n(x) \pm f_n(x) \geq 0$, $g_n(x) \to g(x)$ a.e., and $f_n(x) \to f(x)$ a.e., by Fatou's Lemma

$$\begin{aligned}\int \big(g(x) + f(x)\big)dx \;&\leq\; \liminf_{n\to\infty} \int \big(g_n(x) + f_n(x)\big)dx \\ \int \big(g(x) - f(x)\big)dx \;&\leq\; \liminf_{n\to\infty} \int \big(g_n(x) - f_n(x)\big)dx\end{aligned}$$

Subtract the limit $\int g_n(x)d\mu \to \int g(x)d\mu$ from both inequalities and note that the second inequality is equivalent to $\limsup_{n\to\infty} \int f_n(x)d\mu \leq \int f(x)d\mu$. Then

$$\int f(x)d\mu \;\leq\; \liminf_{n\to\infty} \int f_n(x)d\mu \;\leq\; \limsup_{n\to\infty} \int f_n(x)d\mu \;\leq\; \int f(x)d\mu$$

Since the first and last integrals are the same and finite, $\int f_n(x)d\mu \to \int f(x)d\mu$.

2. (a) If $\int |f_n - f|\, d\mu \to 0$, then

$$\begin{aligned}\left|\int |f_n|d\mu - \int |f|d\mu\right| \;&=\; \left|\int (|f_n| - |f|)d\mu\right| \;\leq\; \int \Big||f_n| - |f|\Big|\, d\mu \\ &\leq\; \int \Big|f_n - f\Big|\, d\mu \to 0\end{aligned}$$

(b) Let $g_n(x) = |f_n(x)| + |f(x)|$ and $g(x) = 2|f(x)|$. Then $|f_n(x) - f(x)| \leq g_n(x)$, $|f_n(x) - f(x)| \to 0$ a.e., $g_n(x) \to g(x)$ a.e., and $\int g_n\, d\mu \to \int g\, d\mu$ since $\int |f_n|\, d\mu \to \int |f|\, d\mu$. Hence, by the previous problem, $\int |f_n - f|\, d\mu \to 0$.

3. It is sufficient to show that $x_n \to x$ implies $F(x_n) \to F(x)$ where

$$F(x) = \int_{-\infty}^{x} f(y)dy = \int I_{(-\infty,x)}(y)f(y)dy$$

Note that $I_{(0,x_n)}(y)f(y) \to I_{(0,x)}(y)f(y)$ for $y \neq x$ (that is, for a.e. y) and

$$|I_{(-\infty,x)}(y)f(y)| \leq |f(y)|$$

since $f \in L^1$ implies $\int |f(y)|dy < \infty$. Thus $F(x_n) \to F(x)$ by the dominated convergence theorem.

4. (Problems 28ac, page 60 in text)

(a) Evaluate $\lim_{n\to\infty}\int_0^\infty \big(1+(x/n)\big)^{-n}\sin(x/n)\,dx$. Proofs I and II below are from student homeworks, in comparison with my more long-winded Proof III. There is also a shorter Proof IV.

In all cases, let $f_n(x)=\big(1+(x/n)\big)^{-n}\sin(x/n)$ and $g_n(x)=\big(1+(x/n)\big)^{-n}$. Then $|f_n(x)|\le g_n(x)$, $f_n(x)\to e^{-x}0=0$ and $g_n(x)\to g(x)=e^{-x}$ for all $x>0$, but $g_n(x)$ is *not* uniformly bounded by e^{-x} since $g_n(x)\sim C_n/x^n \gg e^{-x}$ for fixed n as $x\to\infty$.

In Proofs I–III, the problem is to find dominating function(s) for $g_n(x)$. Proof IV uses a direct estimate of $|f_n(x)|$ and doesn't really use the dominated convergence theorem.

If we can show (i) $g_n(x)\le g(x)$ where $\int_0^\infty g(x)\,dx<\infty$, then $\int_0^\infty f_n\,dx\to 0$ by dominated convergence. If (ii) $\int_0^\infty g_n(x)\,dx\to\int_0^\infty g(x)\,dx<\infty$ where $g_n(x)\to g(x)=e^{-x}$, then we can use Problem 1 to show $\int_0^\infty f_n(x)\,dx\to\int_0^\infty 0\,dx=0$.

Proof I: By the binomial theorem for $x>0, n\ge 4$, $\big(1+(x/n)\big)^n = 1+n(x/n)+\binom{n}{2}(x/n)^2+\ldots \ge 1+((n-1)/2n)x^2$ or even $\ge 1+(n(n-1)(n-2)(n-3)/24)(x/n)^4$. Thus $\big(1+(x/n)\big)^n\ge 1+(1/4)x^2$ or $\ge 1+(1/256)x^4$ for $n\ge 4$. Thus $|\big(1+(x/n)\big)^{-n}\sin(x/n)|\le 4/(4+x^2)$ or $\le 256/(256+x^4)$ and we can use dominated convergence.

Proof II: We use the "moving dominated convergence" result of Problem 1 HW5. Let $f_n(x)$ and $g_n(x)$ be as above. Then $|f_n(x)|\le g_n(x)$, $g_n(x)\to g(x)=e^{-x}$, and $f_n(x)\to e^{-x}(0)=0$ for all x. If we can then show that $\lim_{n\to\infty}\int_0^\infty g_n(x)dx=\int_0^\infty g(x)dx=\int_0^\infty e^{-x}dx=1$, then we can conclude that $\int_0^\infty f_n(x)dx\to 0$.

Now $\int_0^\infty \big(1+(x/n)\big)^{-n}\,dx = n\int_0^\infty (1+x)^{-n}\,dx = n\int_1^\infty x^{-n}\,dx = n/(n-1)$ $\to 1$ as $n\to\infty$. Since $|f_n(x)|\le g_n(x)$, we conclude $\int_0^\infty f_n(x)dx\to 0$.

Proof III: The following argument is longer in this case, but is easier in other cases. I claim that, for fixed $x>0$, $(1+(x/n))^{-n}\downarrow$ as $n\uparrow$. This would imply (for example) that $e^{-x}\le(1+(x/n))^{-n}\le(1+(x/2))^{-2}$ for fixed $x>0$ and $n\ge 2$. The statement is equivalent to $K(y)=(1+xy)^{-1/y}\uparrow$ as $y\uparrow$. Note

$$\frac{d}{dy}\log K(y) \;=\; -\frac{d}{dy}\frac{\log(1+xy)}{y} \;=\; \frac{\log(1+xy)-yx/(1+xy)}{y^2}$$

For $A>0$, $\log(1+A)=\int_1^{1+A}(1/y)dy > A/(1+A)$. Thus, for $A=xy$, we conclude $(d/dy)\log K(y)>0$ for $x>0$, $y>0$. It follows that $(1+(x/n))^{-n}\le(1+x/2)^{-2}$ for $n\ge 2$ and $x>0$. Thus for $n\ge 2$

$$\left|\frac{\sin(x/n)}{\big(1+(x/n)\big)^n}\right| \;\le\; \frac{1}{(1+x/2)^2}$$

which is integrable on $(0,\infty)$. Hence by dominated convergence

$$\lim_{n\to\infty}\int_0^\infty \frac{\sin(x/n)}{\big(1+(x/n)\big)^n}\,dx \;=\; \int_0^\infty \lim_{n\to\infty}\frac{\sin(x/n)}{\big(1+(x/n)\big)^n}\,dx \;=\; 0$$

Proof IV: More directly, if $n \ge 3$, the integral is bounded by

$$\begin{aligned}
\int \left|\big((1+(x/n)\big)^{-n}\sin(x/n)\right|dx &\le \int_0^\infty \big((1+(x/n)\big)^{-n}(x/n)\,dx\\
&= n\int_0^\infty (1+x)^{-n}x\,dx = n\int_1^\infty x^{-n}(x-1)\,dx\\
&= n\int_1^\infty \left(x^{-n+1}-x^{-n}\right)dx = n\left(\int_1^\infty x^{-n+1}\,dx - \int_1^\infty x^{-n}\,dx\right)\\
&= n\left(\frac{1}{n-2}-\frac{1}{n-1}\right) \;=\; \frac{n}{n-1)(n-2)} \;\to\; 0
\end{aligned}$$

(c) This integral is $\int_0^\infty \frac{\sin(x/n)}{x/n}\frac{dx}{1+x^2}dx$ where, since $|\sin(x)/x| \le 1$,

$$\left|\frac{\sin(x/n)}{x/n}\frac{1}{1+x^2}\right| \;\le\; \frac{1}{1+x^2}$$

which is integrable on $(0,\infty)$. Thus by dominated convergence

$$\lim_{n\to\infty}\int_0^\infty \frac{\sin(x/n)}{x/n}\frac{dx}{1+x^2} \;=\; \int_0^\infty \lim_{n\to\infty}\frac{\sin(x/n)}{x/n}\frac{dx}{1+x^2} \;=\; \int_0^\infty \frac{dx}{1+x^2} \;=\; \frac{\pi}{2}$$

5. (Problem 31ac page 60)
(a) For $a > 0$,

$$\int_{-\infty}^\infty e^{-x^2}\cos(ax)dx \;=\; \int_{-\infty}^\infty e^{-x^2}\sum_{n=0}^\infty \frac{(-1)^n(ax)^{2n}}{(2n)!}\,dx \;=\; \int_{-\infty}^\infty e^{-x^2}\sum_{n=0}^\infty f_n(x)\,dx$$

where $f_n(x) = (-1)^n(ax)^{2n}/(2n!)$. The problem is to justify interchanging the integral and the infinite sum. Note

$$\begin{aligned}
\int_{-\infty}^\infty e^{-x^2}\sum_{n=0}^\infty |f_n(x)|\,dx &= \int_{-\infty}^\infty e^{-x^2}\sum_{n=0}^\infty \frac{(ax)^{2n}}{(2n)!}\,dx\\
&= \int_{-\infty}^\infty e^{-x^2}\frac{(e^{ax}+e^{-ax})}{2}\,dx \;<\; \infty
\end{aligned}$$

Thus, by a form of dominated convergence (see Theorem 2.25 in the text), we can interchange the sum and the integral and obtain

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n\frac{a^{2n}}{(2n)!}\int_{-\infty}^{\infty}e^{-x^2}x^{2n}dx &= \sum_{n=0}^{\infty}(-1)^n\frac{a^{2n}}{(2n)!}\frac{(2n)!\sqrt{\pi}}{4^n\,n!}\\ &= \sqrt{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!}\left(\frac{a^2}{4}\right)^n = \sqrt{\pi}\,e^{-a^2/4}\end{aligned}$$

(c) For $a > 1$, since the terms in the sum below are nonnegative, and by a form of the monotone convergence theorem (see Theorem 2.15 in the text)

$$\begin{aligned}\int_0^{\infty}x^{a-1}\frac{1}{e^x-1}\,dx &= \int_0^{\infty}x^{a-1}\frac{e^{-x}}{1-e^{-x}}\,dx = \int_0^{\infty}x^{a-1}\sum_{n=1}^{\infty}e^{-nx}dx\\ = \sum_{n=1}^{\infty}\int_0^{\infty}x^{a-1}e^{-nx}dx &= \sum_{n=1}^{\infty}\frac{1}{n^a}\int_0^{\infty}x^{a-1}e^{-x}dx = \Gamma(a)\zeta(a)\end{aligned}$$