Ma 5051 — Real Variables and Functional Analysis

Problem Set #7 — Due October 29, 2009

Prof. Sawyer — Washington University

Unless otherwise said, all measures (signed or positive) are defined on $(X, \mathcal{M})$. If ρ, ν are signed measures, then $\rho \leq \nu$ means that $\rho(A) \leq \nu(A)$ for all $A \in \mathcal{M}$.

1. (Problem 3bc, page 88) Let ν be a signed measure. (See the conventions in the heading.) Recall that we defined $L^1(\nu) = L^1(|\nu|)$ in class. Show that

(a) For all $f \in L^1(\nu)$, $\;|\int f(x)\,d\nu\,| \leq \int |f|\,d|\nu|$

(b) For all $E \in \mathcal{M}$, $|\nu|(E) = \sup\{\,|\int_E f(x)\,d\nu| : |f| \leq 1\,\}$

2. (Problem 4, page 88) If ν is a signed measure such that $\nu = \lambda - \mu$ for positive measures λ and μ, then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$. (See the proof of the Jordan decomposition theorem on page 87 for the definitions of $\nu^{\pm}$)

3. (Like Problem 6, page 88) Let ν be the signed measure defined by $\nu(A) = \int_A f(x)\,d\mu$ for $A \in \mathcal{M}$ where $f \in L^1(\mu)$ for a positive measure μ. Describe the Hahn decomposition of ν and the positive, negative, and total variations of ν (that is, ν^+, ν^-, and ν) in terms of f and μ.

4. (Problem 7, page 88) Let ν be a signed measure and $E \in \mathcal{M}$. Prove that

(a) $\nu^+(E) = \sup\{\,\nu(F) : F \in \mathcal{M},\ F \subseteq E\,\}$

(b) $\nu^-(E) = -\inf\{\,\nu(F) : F \in \mathcal{M},\ F \subseteq E\,\}$

(c) $|\nu|(E) = \sup\{\,\sum_{j=1}^n |\nu(E_j)| : n \geq 1,\ E \subseteq \bigcup_{j=1}^n E_j$ for disjoint $E_j \in \mathcal{M}\,\}$

(Note: Extra condition added in part (c) to match the textbook.)

5. (Problem 9, page 92) Let μ and $\{\,\nu_j : j = 1, 2, \ldots\}$ be positive measures on $(X, \mathcal{M})$. Show that if $\nu_j \perp \mu$ for all j, then $\sum_{j=1}^{\infty} \nu_j \perp \mu$. Show also that if $\nu_j \ll \mu$ for all j, then $\sum_{j=1}^{\infty} \nu_j \ll \mu$.