Ma 5051 — Real Variables and Functional Analysis

Problem Set #8 — Due November 12, 2009

Prof. Sawyer — Washington University

Let $(X, \mathcal{M})$ be a measurable space. Unless otherwise said, all measures (signed, positive, or complex) are defined on $(X, \mathcal{M})$.

1. (Problem 12, page 92) Let $\nu_1, \mu_1, \nu_2, \mu_2$ be σ-finite positive measures such that $\nu_1 \ll \mu_1$ and $\nu_2 \ll \mu_2$. Show that $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ on $X \times X$ and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) \;=\; \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2) \qquad \text{for } (\mu_1 \times \mu_2)\text{-a.e. } (x_1, x_2)$$

2. (Like Problem 16, page 92) Let μ, ν be σ-finite positive measures with $\nu \ll \mu$. Let $\lambda = \mu + \nu$. Show that

(a) $\nu \ll \lambda$ and $\lambda \ll \mu$

(b) if $f(x) = (d\nu/d\lambda)(x)$, then $0 \le f(x) < 1$, and

(c) $(d\nu/d\mu)(x) = f(x)/\big(1 - f(x)\big)$.

3. (Like Problem 17, page 93) Let μ be a finite positive measure. Define $\nu(E) = \int_E f(x) d\mu$ for some $f \in L^1(X, \mathcal{M}, \mu)$. Let $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{M}$. Show that, even though $f(x)$ may not be $\mathcal{A}$-measurable, there exists some $\mathcal{A}$-measurable $g \in L^1(\mu)$ such that

$$\nu(E) = \int_E f(x) d\mu = \int_E g(x) d\mu \qquad \text{for all } E \in \mathcal{A} \tag{1}$$

and, moreover, that $g(x)$ is μ-a.e. unique: That is, if $g_1(x)$ and $g_2(x)$ are two $\mathcal{A}$-measurable functions satisfying (1), then $g_1(x) = g_2(x)$ μ-a.e. (The $\mathcal{A}$-measurable function $g(x)$ is called the *conditional expectation* of $f(x)$ given $\mathcal{A}$.)

4. (Problem 20, page 94) Show that if ν is a complex measure on $(X, \mathcal{M})$ with $\nu(X) = |\nu|(X)$, then $\nu = |\nu|$.

5. (Similar to Problem 21, page 94) If ν is a complex measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$, show that

$$|\nu|(E) \;=\; \sup \left\{ \left| \int_E f(x) d\nu \right| \;:\; |f(x)| \le 1 \text{ on } E \right\}$$