1. Let $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$, where $\mathcal{M}_1 \otimes \mathcal{M}_2$ is the product $\sigma$-algebra. By assumption, $\nu_1(A) = \int_A g_1(x)\mu_1(dx)$ for $A \in \mathcal{M}_1$ and $\nu_2(B) = \int_B g_2(y)\mu_2(dy)$ for $B \in \mathcal{M}_1$ where $g_1(x) = (dv_1/d\mu_1)(x) \geq 0$ a.e. ($\mu_1$) and $g_2(y) = (dv_1/d\mu_2)(y) \geq 0$ a.e. ($\mu_2$) since $\nu_1, \nu_2, \mu_1, \mu_2$ are positive measures. Thus by Tonelli’s Theorem (page 67)

$$(\nu_1 \times \nu_2)(E) = \int_{X_1 \times X_2} I_E(x_1, x_2)\,d(\nu_1 \times \nu_2)(x_1, x_2)$$

$$(1)$$

$$= \int_{X_1} \left( \int_{X_2} I_E(x_1, x_2)\,d\nu_2(x_2) \right)\,d\nu_1(x_1)$$

$$= \int_{X_1} \left( \int_{X_2} I_E(x_1, x_2)g_2(x_2)\,d\mu_2(x_2) \right)\,g_1(x_1)\,d\mu_1(x_1)$$

$$= \int_{X_1} \left( \int_{X_2} I_E(x_1, x_2)g_1(x_1)g_2(x_2)\,d\mu_2(x_2) \right)\,d\mu_1(x_1)$$

and by Tonelli’s Theorem a second time

$$(\nu_1 \times \nu_2)(E) = \int_{X_1 \times X_2} I_E(x_1, x_2)g_1(x_1)g_2(x_2)\,d(\mu_1 \times \mu_2)(x_1, x_2)$$

$$(2)$$

This implies that $(\nu_1 \times \nu_2) \ll (\mu_1 \times \mu_2)$ and the Radon-Nikodym derivative is $d(\nu_1 \times \nu_2)/d(\mu_1 \times \mu_2) = g_1(x_1)g_2(x_2) = (dv_1/d\mu_1)(x_1)(dv_2/d\mu_2)(x_2)$.

**Remark.** Since all these measures are $\sigma$-finite, it is sufficient to verify (2) on a generating semi-ring of sets $E$. Since the set of measurable rectangles $\{ A \times B : A \in \mathcal{M}_1, B \in \mathcal{M}_2 \}$ is a generating semi-ring for $\mathcal{M}_1 \otimes \mathcal{M}_2$, it is sufficient to assume that $E = A \times B$ in (1) is a measurable rectangle. This simplifies the proof slightly.

2. (a) If $\lambda(E) = \nu(E) + \mu(E) = 0$, then $\nu(E) = \mu(E) = 0$ and $\nu \ll \lambda$. If $\mu(E) = 0$, then $\nu(E) = 0$ since $\nu \ll \mu$ and hence $\lambda(E) = \nu(E) + \mu(E) = 0$, which implies $\lambda \ll \mu$.

(b) Assume $\nu(E) = \int_E g\,d\mu$. Since $\nu$ and $\mu$ are positive and $\sigma$-finite, $0 \leq g(x) < \infty$ a.e. $\mu$. Then $\lambda(E) = \int_E (1 + g)\,d\mu$. Hence $\int h\,d\lambda = \int h(1 + g)\,d\mu$ for all measurable $h(x) \geq 0$. Setting $h = (1 + g)^{-1}I_E$, $\mu(E) = \int_E (1 + g)^{-1}\,d\lambda$ and

$$\nu(E) = \int_E g\,d\mu = \int_E \frac{g}{1 + g}\,d\lambda = \int_E f\,d\lambda$$
for \( f = d\nu/d\lambda = g/(1 + g) \). Since \( 0 \leq g < \infty \), \( 0 \leq f(x) < 1 \) a.e. \( \mu \).

(c) \( f = g/(1+g) \) implies \( f + fg = g \) and \( g = f/(1-f) \), so \( d\nu/d\mu = g = f/(1-f) \).

3. Here \( \mu, \nu \) are finite positive measures on \((X, \mathcal{M})\) with \( \nu(E) = \int f \, d\mu \) for some \( \mathcal{M}\)-measurable functions \( f(x) \geq 0 \). Restrict both measures \( \mu, \nu \) to \((X, \mathcal{A})\) for a \( \sigma\)-algebra \( \mathcal{A} \subseteq \mathcal{M} \). It may not be true that \( \nu(E) = \int_E f \, d\mu \) for \( f \in L^1(X, \mathcal{A}, \mu) \), but \( \mu(E) = 0 \) for \( E \in \mathcal{A} \) still implies \( \nu(E) = 0 \) since \( \mathcal{A} \subseteq \mathcal{M} \). It then follows from the Radon-Nikodym theorem that \( \mu(E) = \int_E g \, d\mu \) for \( E \in \mathcal{A} \) and \( g \in L^1(X, \mathcal{A}, \mu) \), where now we are guaranteed that \( g(x) \) is \( \mathcal{A}\)-measurable.

If \( \mu(E) = \int_E g_1 \, d\mu = \int_E g_2 \, d\mu \) for all \( E \in \mathcal{A} \) and \( g_1, g_2 \in L^1(X, \mathcal{A}, \mu) \), then \( \int_E (g_1 - g_2) \, d\mu = 0 \) for all \( E \in \mathcal{A} \) and by standard arguments for \((X, \mathcal{A}, \mu)\), \( g_1(x) = g_2(x) \) a.e. \( \mu \).

4. Assume \( d\nu = f \, d\mu \) for some positive measure \( \mu \) where \( f \in L^1(X, \mathcal{M}, \mu) \). Then (by definition) \( d|\nu| = |f| \, d\mu \) and \( \nu(X) = \int f(x) \, d\mu = |\nu|(X) = \int |f(x)| \, d\mu = 0 \). Thus \( \int (|f(x)| - f(x)) \, d\mu = 0 \) and

\[
\text{Re} \int (|f(x)| - f(x)) \, d\mu = \int (|f(x)| - \text{Re} f(x)) \, d\mu = 0
\]

In general, \( \int g(x) \, d\mu(x) = 0 \) for \( g(x) \geq 0 \) and a positive measure \( \mu \) implies \( g(x) = 0 \) a.e. \( \mu \). Since \( g(x) = |f(x)| - \text{Re} f(x) \geq 0 \) for all \( x \), it follows that \( \text{Re} f(x) = |f(x)| \) a.e. \( \mu \). Since \( |f(x)| = |\text{Re} f(x) + i\text{Im} f(x)| \), this also implies \( \text{Im} f(x) = 0 \) a.e. \( \mu \). Thus \( f(x) = |f(x)| \) a.e. \( \mu \) and \( \nu = |\nu| \).

5. Assume \( \nu(E) = \int_E g(x) \, d\mu \) for complex \( g \in L^1(X, \mathcal{M}, \mu) \). Then (by definition) \( |\nu|(E) = \int_E |g(x)| \, d\mu \) and it is sufficient to prove

\[
\int_E |g| \, d\mu = \sup \left\{ \left| \int_E f(x) g(x) \, d\mu \right| : |f(x)| \leq 1 \text{ on } E \right\}
\]  

(3)

Since \( \int_E f g \, d\mu \leq \int_E |f g| \, d\mu \leq \int_E |g| \, d\mu \) for complex functions \( f, g \) with \( |f(x)| \leq 1 \), the right-hand side of (3) is less than or equal to the left-hand side. Define \( f(x) = |g(x)|/g(x) \) if \( g(x) \neq 0 \) and \( f(x) = 0 \) if \( g(x) = 0 \). Then \( \int_E f(x) g(x) \, d\mu = \int_E |g(x)| \, d\mu \) and the two sides of (3) are identical.