1. Let $\nu = \mu_1 \times \mu_2 = (X_1 \times X_2, \mathcal{M}_1 \otimes \mathcal{M}_2, \mu_1 \times \mu_2)$ be the product measure of two $\sigma$-finite measures $(X_i, \mathcal{M}_i, \mu_i)$ $(i = 1, 2)$. Let $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$ be a measurable subset of $X_1 \times X_2$ with $\nu(E) > 0$. Prove that, for any constant $\theta$ with $0 < \theta < 1$, there exists a measurable rectangle $B = A_1 \times A_2 \in \mathcal{M}_1 \times \mathcal{M}_2$ such that $\nu(B) < \infty$ and $\nu(E \cap B) > \theta \nu(B)$.

2. Let $T f(x) = \int_0^1 K(x, y) f(y) dy$ for $(x, y) \in R \times I$ for $I = [0, 1]$, $f \in L^1(I, B, m)$ for Lebesgue measure $m$ $(m(dy) = dy)$, and $|K(x, y)| \leq Ce^{-|x-y|}$ for all $(x, y)$ for some constant $C < \infty$.

Assume $\lim_{n \to \infty} \int_0^1 |f_n(x) - f(x)| dx = 0$ for $f_n, f \in L^1(I)$ and set $g_n(x) = Tf_n(x)$ and $g(x) = Tf(x)$. Prove that

(a) $g_n(x) \to g(x)$ for all $x \in R$

(b) $\lim_{n \to \infty} \int_{-\infty}^\infty |g_n(y) - g(y)| dy = 0$.

3. Define $f(x, y)$ on $R^2$ by $f(x, y) = \sin(x - y)$ for $x, y$ in the strip $S = \{(x, y) : y \leq x \leq y + 2\pi\}$ and $f(x, y) = 0$ for $(x, y) \notin S$. Calculate the two iterated integrals $I_1 = \int_0^\infty \int_0^\infty f(x, y) dx dy$ and $I_2 = \int_0^\infty \int_0^\infty f(x, y) dy dx$. If you conclude $I_1 \neq I_2$, explain how this is possible since Lebesgue measure on $[0, \infty)$ is $\sigma$-finite. Justify your reasoning.

4. Let $f_1(x), f_2(x)$ be two measurable functions on a measurable space $(X, \mathcal{M})$. Let $\mathcal{A} = \mathcal{B}(f_1, f_2)$ be the smallest $\sigma$-algebra $\mathcal{A} \subseteq \mathcal{M}$ with respect to which both $f_1(x)$ and $f_2(x)$ are measurable. (That is, $\mathcal{A}$ is the $\sigma$-algebra generated by the sets $\{x : f_i(x) \leq \lambda_i\} \subseteq X$ for $\lambda_i \in R$ and $i = 1, 2$.)

Show that any nonnegative $\mathcal{A}$-measurable function $g(x)$ on $X$ can be written $g(x) = \phi(f_1(x), f_2(x))$ for some Borel function $\phi(y_1, y_2)$ on $R^2$. (Hint: Prove this first for $g(x) = I_E(x)$ for $E \in \mathcal{A}$ and approximate a general $\mathcal{A}$-measurable function $g(x)$ by simple functions. Show that $h(x) = (f_1(x), f_2(x)) : X \to R^2$ is a measurable mapping in the sense of Section 2.1 in the text.)

5. Define

$$||f||_\alpha = \sup_{x \in X} |f(x)| + \sup_{x, y \in X, y \neq x} \frac{|f(y) - f(x)|}{d(x, y)^\alpha}$$
where \((X, d)\) is a compact metric space and \(0 < \alpha < 1\). (An earlier version of this problem used the symbol \(\rho\) for both metrics.)

(a) \((1/4)\) Prove that \(\rho(f, g) = \|f - g\|_\alpha\) is a metric on the space

\[ C^\alpha(X) = \{ f \in C(X) : \|f\|_\alpha < \infty \} \]

(b) \((1/4)\) Prove that if \(f_n, f \in C(X)\) satisfy \(f_n(x) \to f(x)\) uniformly on \(X\), then

\[ \|f\|_\alpha \leq \liminf_{n \to \infty} \|f_n\|_\alpha \]

(c) \((1/2)\) Let \(K = \{ f \in C(X) : \|f\|_\alpha \leq M \}\) for a constant \(M < \infty\). Prove that \(K\) is a compact subset of \(C(X)\).

6. Let \(\{ X_\alpha : \alpha \in A \}\) be a family of Hausdorff topological spaces \(X_\alpha = (X_\alpha, T_\alpha)\) for which infinitely many of the spaces \(X_\alpha\) are not compact. Prove that every compact subset \(K\) of the product space \(X = \prod \{ X_\alpha : \alpha \in A \}\) is nowhere dense in the product topology.

7. Let \(X = (X, T)\) be a topological space that is not Hausdorff. Prove that there exists a net \(\{ x_i : i \in I \}\) in \(X\) that converges to two distinct points. \((Hint: \) Try \(I = N(x) \times N(y)\) for points \(x, y \in X\) where \(N(x), N(y)\) are neighborhood bases for \(x\) and \(y\), respectively. Be sure to specify the partial ordering on \(I\) and show that it is directed.)

8. Let \(X = (X, T)\) be a topological space. Let \(\{ x_i : i \in I \}\) be a net in \(X\), and let \(\{ y_j : j \in J \}\) be a subnet of \(\{ x_i \}\).

(a) Prove that, for all \(i_0 \in I\), there exists \(J_0 \in J\) such that

\[ \{ y_j : j \in J, j \geq j_0 \} \subseteq \{ x_i : i \in I, i \geq i_0 \} \]

\((Hint: \) This was proven in class, but give a (short) proof.)

(b) Prove that any cluster point of \(\{ y_j : j \in J \}\) is also a cluster point of \(\{ x_i : i \in I \}\).

\((Hint: \) You can use the result of Problem 33 p127 in the text, which was proven as a theorem in class.)

(c) Prove that if \(x_i \to x\) for some \(x \in X\), then also \(y_j \to x\).