Ma 5051 — Real Variables and Functional Analysis

Take-Home Final — Due December 17, 2009

Prof. Sawyer — Washington University

Problems in text are from G. B. Folland, *Real Analysis: Modern techniques and their applications*. Eight problems on two pages.

1. Let $\nu = \mu_1 \times \mu_2 = (X_1 \times X_2, \mathcal{M}_1 \otimes \mathcal{M}_2, \mu_1 \times \mu_2)$ be the product measure of two σ-finite measures $(X_i, \mathcal{M}_i, \mu_i)$ $(i = 1, 2)$. Let $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$ be a measurable subset of $X_1 \times X_2$ with $\nu(E) > 0$. Prove that, for any constant θ with $0 < \theta < 1$, there exists a measurable rectangle $B = A_1 \times A_2 \in \mathcal{M}_1 \times \mathcal{M}_2$ such that $\nu(B) < \infty$ and $\nu(E \cap B) > \theta\nu(B)$.

2. Let $Tf(x) = \int_0^1 K(x,y)f(y)dy$ for $(x,y) \in R \times I$ for $I = [0,1]$, $f \in L^1(I, \mathcal{B}, m)$ for Lebesgue measure m $(m(dy) = dy)$, and $|K(x,y)| \le Ce^{-|x-y|}$ for all (x,y) for some constant $C < \infty$.

Assume $\lim_{n\to\infty} \int_0^1 |f_n(x) - f(x)|\,dx = 0$ for $f_n, f \in L^1(I)$ and set $g_n(x) = Tf_n(x)$ and $g(x) = Tf(x)$. Prove that

(a) $g_n(x) \to g(x)$ for all $x \in R$

(b) $\lim_{n\to\infty} \int_{-\infty}^{\infty} |g_n(y) - g(y)|dy = 0$.

3. Define $f(x,y)$ on R^2 by $f(x,y) = \sin(x-y)$ for x, y in the strip $S = \{\,(x,y) : y \le x \le y + 2\pi\,\}$ and $f(x,y) = 0$ for $(x,y) \notin S$. Calculate the two iterated integrals $I_1 = \int_0^\infty \int_0^\infty f(x,y)dxdy$ and $I_2 = \int_0^\infty \int_0^\infty f(x,y)dydx$. If you conclude $I_1 \ne I_2$, explain how this is possible since Lebesgue measure on $[0,\infty)$ is σ-finite. Justify your reasoning.

4. Let $f_1(x), f_2(x)$ be two measurable functions on a measurable space $(X, \mathcal{M})$. Let $\mathcal{A} = \mathcal{B}(f_1, f_2)$ be the smallest σ-algebra $\mathcal{A} \subseteq \mathcal{M}$ with respect to which both $f_1(x)$ and $f_2(x)$ are measurable. (That is, $\mathcal{A}$ is the σ-algebra generated by the sets $\{\,x : f_i(x) \le \lambda_i\,\} \subseteq X$ for $\lambda_i \in R$ and $i = 1, 2$.)

Show that any nonnegative $\mathcal{A}$-measurable function $g(x)$ on X can be written $g(x) = \phi\big(f_1(x), f_2(x)\big)$ for some Borel function $\phi(y_1, y_2)$ on R^2. (*Hint*: Prove this first for $g(x) = I_E(x)$ for $E \in \mathcal{A}$ and approximate a general $\mathcal{A}$-measurable function $g(x)$ by simple functions. Show that $h(x) = (f_1(x), f_2(x)) : X \to R^2$ is a measurable mapping in the sense of Section 2.1 in the text.)

5. Define

$$\|f\|_\alpha \;=\; \sup_{x\in X} |f(x)| \;+\; \sup_{x,y\in X, y\ne x} \frac{|f(y) - f(x)|}{d(x,y)^\alpha}$$

where (X, d) is a compact metric space and $0 < \alpha < 1$. (An earlier version of this problem used the symbol ρ for both metrics.)

(a) (1/4) Prove that $\rho(f, g) = \|f - g\|_\alpha$ is a metric on the space

$$C^\alpha(X) = \{ f \in C(X) : \|f\|_\alpha < \infty \}$$

(b) (1/4) Prove that if $f_n, f \in C(X)$ satisfy $f_n(x) \to f(x)$ uniformly on X, then

$$\|f\|_\alpha \; \le \; \liminf_{n\to\infty} \|f_n\|_\alpha$$

(c) (1/2) Let $K = \{ f \in C(X) : \|f\|_\alpha \le M \}$ for a constant $M < \infty$. Prove that K is a compact subset of $C(X)$.

6. Let $\{ X_\alpha : \alpha \in A \}$ be a family of Hausdorff topological spaces $X_\alpha = (X_\alpha, \mathcal{T}_\alpha)$ for which infinitely many of the spaces X_α are not compact. Prove that every compact subset K of the product space $X = \prod\{ X_\alpha : \alpha \in A \}$ is nowhere dense in the product topology.

7. Let $X = (X, \mathcal{T})$ be a topological space that is not Hausdorff. Prove that there exists a net $\{ x_i : i \in I \}$ in X that converges to two distinct points. (*Hint*: Try $I = \mathcal{N}(x) \times \mathcal{N}(y)$ for points $x, y \in X$ where $\mathcal{N}(x), \mathcal{N}(y)$ are neighborhood bases for x and y, respectively. Be sure to specify the partial ordering on I and show that it is directed.)

8. Let $X = (X, \mathcal{T})$ be a topological space. Let $\{ x_i : i \in I \}$ be a net in X, and let $\{ y_j : j \in J \}$ be a subnet of $\{ x_i \}$.

(a) Prove that, for all $i_0 \in I$, there exists $J_0 \in J$ such that

$$\{ y_j : j \in J, j \ge j_0 \} \;\; \subseteq \;\; \{ x_i : i \in I, i \ge i_0 \}$$

(*Hint*: This was proven in class, but give a (short) proof.)

(b) Prove that any cluster point of $\{ y_j : j \in J \}$ is also a cluster point of $\{ x_i : i \in I \}$.

(*Hint*: You can use the result of Problem 33 p127 in the text, which was proven as a theorem in class.)

(c) Prove that if $x_i \to x$ for some $x \in X$, then also $y_j \to x$.