Let $(X, \mathcal{M}, \mu)$ be a measure space. Recall $\int_A f(x) \, d\mu = \int I_A(x) f(x) \, d\mu$ for $A \in \mathcal{M}$ and $f \in L^+$, where $I_A(x)$ is the indicator function of $A$.

1. (a) Evaluate the limit and justify the calculation:
   \[
   \lim_{n \to \infty} \int_0^n x^k (1 - n^{-1} x)^n \, dx
   \]

   (b) Evaluate the limit and justify the calculation:
   \[
   \lim_{n \to \infty} \sqrt{n} \int_0^1 (1 + nx^2)(1 + x^2)^{-n} \, dx
   \]

2. Expand the following as a power series in $a$ and justify any interchange of integrals and sums:
   \[
   \int_0^\infty e^{-y^2} \sin(ay) \, dy
   \]

3. (Like Problem 33 page 63) If $f_n(x) \geq 0$ are measurable and $f_n \to f$ in measure for a measure $\mu$, show that $\int f(x) \, d\mu \leq \liminf_{n \to \infty} \int f_n(x) \, d\mu$.

4. For $X = [0, 1]$, let $\mu = m + 2\delta_a$ for some $a \in (0, 1)$ where $m$ is Lebesgue measure and $\delta_a$ is the delta measure based at $a$. (That is, $\delta_a(E) = 1$ if $a \in E$ and $\delta_a(E) = 0$ if $a \notin E$.) Thus $\mu(X) = 3$. Evaluate the following limit and justify your calculations:
   \[
   \lim_{n \to \infty} \int_0^1 \int_0^1 e^{-n(x-y)^2} \, d\mu(x) \, d\mu(y)
   \]
   \text{(Warning: The measure $\mu(A)$ may have atoms.)}

5. (Like Problem 50 page 69) Assume that $(X, \mathcal{M}, \mu)$ is a $\sigma$-finite measure and assume $f \in L^+(X, \mathcal{M})$. Let $m$ be Lebesgue measure on $R^+ = [0, \infty)$ and set
   \[
   G_f = \{(x, y) \in (X \times [0, \infty)) : 0 \leq y < f(x)\}
   \]

   Use Theorem 2.10 page 47 to show that $G_f \in \mathcal{M} \otimes \mathcal{B}(R^+)$. Also, show that
   \[
   (\mu \times m)(G_f) = \int_X f(x) \, d\mu = \int_0^\infty \mu(\{ f > y \}) \, dy
   \]
   \text{(This shows that the integral can be viewed as “the area under the curve”.)}
6. (Like Problem 51 page 69) Let \((X, \mathcal{M}_1, \mu)\) and \((Y, \mathcal{M}_2, \beta)\) be two arbitrary measure spaces, not necessarily \(\sigma\)-finite (so that you cannot apply Fubini’s theorem directly). Show

(a) If \(f : X \to \mathbb{R}\) is \(\mathcal{M}_1\)-measurable and \(g : Y \to \mathbb{R}\) is \(\mathcal{M}_2\)-measurable for the real numbers \(\mathbb{R}\), then \(h(x, y) = f(x)g(y)\) is \((\mathcal{M}_1 \otimes \mathcal{M}_2)\)-measurable.

(b) If \(f \in L^1(\mu)\) and \(g \in L^1(\beta)\), then \(h \in L^1(\mu \times \beta)\) and

\[
\int h(z)d(\mu \times \beta)(z) = \left(\int f(x)d\mu(x)\right)\left(\int g(y)d\beta(y)\right)
\]

7. (Like Problem 54 page 77) Theorem 2.44 has four statements about invertible \(n \times n\) matrices \(T\). Replacing “Lebesgue measurable function” by “Borel function” and \(L^n\) by \(\mathcal{B}(\mathbb{R}^n)\), which of the four statements are true or false if \(T\) is not invertible?

8. (Like Problem 61 page 77) Define

\[
I_{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1}f(y)\,dy
\]

for \(\alpha > 0, x \in (0, \infty),\) and \(f(x)\) continuous on \([0, \infty)\). The functional \(I_{\alpha}f(x)\) is called the \(\alpha^{th}\) fractional integral of \(f(x)\). Show that

(a) \(I_{\alpha+\beta}f(x) = I_{\alpha}(I_{\beta}f)(x)\) for all \(\alpha, \beta > 0\) and \(x > 0\). (You can use the identity in Problem 60 page 77 in the text.)

(b) If \(n\) is an integer with \(n \geq 1\), then \((d/dx)^n I_{\alpha}f(x) = f(x)\) for \(x > 0\). (In this sense, \(I_{n}f(x)\) is an \(n\)-fold integral of \(f(x)\).)