1. Let $\Omega = \{0, 1, 2, \ldots\}$ be the nonnegative integers and let $\mathcal{A}$ be the set of all finite subsets of $\Omega$. Show that

(i) $\mathcal{A}$ is a semi-ring of subsets of $\Omega$.

(ii) $\mathcal{B}(\mathcal{A}) = 2^\Omega$, where $\mathcal{B}(\mathcal{A})$ is the smallest $\sigma$-algebra of subsets of $\Omega$ that contains $\mathcal{A}$ and $2^\Omega$ means the set of all subsets of $\Omega$.

(iii) There does not exist a probability measure $(\Omega, \mathcal{F}, P)$ for $\mathcal{F} = \mathcal{B}(\mathcal{A})$ such that $\mu(\{\{n\}\}) = \alpha$ has the same value for all $n \in \Omega$.

2. Let $X \geq 0$ be a nonnegative random variable (r.v.) on a probability space $(\Omega, \mathcal{F}, P)$ and assume that $r > 0$. Show that

$$E(X^r) = \int_0^\infty ry^{r-1}P(X \geq y)\,dy$$

Conclude that $E(X) < \infty$ if and only if

$$\int_0^\infty P(X \geq y)\,dy < \infty$$

(Hint: Write $E(X^r) = E(\int_0^{X^r} du)$, express in terms of a product measure $\mu_2(d\omega dy)$ on $\Omega \times [0, \infty)$, and use Fubini’s Theorem.)

3. Let $X$ be a uniformly distributed r.v. on a probability space $(\Omega, \mathcal{F}, P)$. (That is, $P(X \leq x) = x$ for $0 \leq x \leq 1$.) Let $Y = r \log(1/X)$ for some $r > 0$. Prove that $Y$ has a density function $f_Y(y)$ with respect to Lebesgue measure and find $f_Y(y)$. (Hint: This holds if and only if $E(\phi(Y)) = \int_0^\infty \phi(y)f_Y(y)\,dy$ for all bounded Borel functions $\phi(x) \geq 0$.)

4. Let $\{X_n\}$ be an independent and identically distributed sequence of random variables on a probability space $(\Omega, \mathcal{F}, P)$ such that $X_i \geq 0$ and $E(X_i) = \infty$. Use
the strong law of large numbers for independent random variables with $E(X^4) < \infty$
to prove that
\[ \lim_{n \to \infty} \frac{X_1 + X_2 + \cdots + X_n}{n} = \infty \quad \text{a.s.} \]

(Hint: Consider $X_i^C = \min\{X_i, C\}$ for $C > 0$.)

5. Let $X$ and $Y$ be independent and identically distributed random variables. Find the limit
\[ \lim_{n \to \infty} \exp\left(e^{-n(X-Y)^2}\right) \]

(Hint: Express the expectation in terms of the joint distribution function of $X$ and $Y$. If you take a limit inside an integral, explain why it is valid. Be careful!)

6. Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be two sets of independent random variables on a probability space $(\Omega, \mathcal{F}, P)$. Assume that the $X_i$ have a density $f(x) = f_X(x)$ on $\mathbb{R}$ and the $Y_j$ have density $g(y) = f_Y(y)$, respectively. That is,
\[
P(X_i \leq x) = F_X(x) = \int_0^x f(u) du, \quad \text{all } i, \text{ all } x \\
P(Y_j \leq y) = F_Y(y) = \int_0^y g(v) dv, \quad \text{all } j, \text{ all } y
\]
Assume $g(y) > 0$ for all $y \in \mathbb{R}$ and define
\[
L_R(y_1, y_2, \ldots, y_n) = \frac{f(y_1)f(y_2)\cdots f(y_n)}{g(y_1)g(y_2)\cdots g(y_n)}
\]
Prove that
\[
E\left(\phi(Y_1, Y_2, \ldots, Y_n)L_R(Y_1, \ldots, Y_n)\right) = E\left(\phi(X_1, X_2, \ldots, X_n)\right)
\]
for all nonnegative bounded continuous functions $\phi(y_1, \ldots, y_n)$ on $\mathbb{R}^n$. (Hint: Use the fact that $X_i$ and $Y_j$ are independent and write out both sides as integrals on $\mathbb{R}^n$.)