Solutions to HW4

§2.4 1a)

\[ v = \int \frac{1}{y_1^2} e^{-\int p(x) \, dx} \cdot \frac{1}{\sin^2 x} e^{-\int 0 \, dx} \, dx = \int \frac{1}{\sin^2 x} e^{-1} \, dx = e^{-1} \csc^2 x \]

So \( v = -e^{-1} \cot x \). And

\[ y_2 = v y_1 = -e^{-1} \cot x \cdot \sin x = -e^{-1} \cos x. \]

The general solution is

\[ y = A \sin x + B \cos x. \]

2. \( v = \int \frac{1}{y_1^2} e^{-\int p(x) \, dx} \cdot \frac{1}{1^2} e^{-\int 3 \, dx} \, dx = \int x^{-3} \, dx = -\frac{1}{2} x^{-2}. \)

So \( y_2 = v y_1 = -\frac{1}{2} x^{-2}. \) The general sol. is

\[ y = A - 1 + B \cdot x^{-2} = A + Bx^{-2}. \]

6 b) \( v = \int \frac{1}{y_1^2} e^{-\int p(x) \, dx} \cdot \frac{1}{x^2} e^{-\int 2 \, dx} \, dx = \int \frac{1}{x^2} \cdot x^{-3} \, dx = -\frac{1}{3} x^{-3}. \)

So \( y_2 = v y_1 = -\frac{1}{3} x^{-3}, x = -\frac{1}{3} x^{-2}. \)
The general solution is

\[ y = A e^x + B x^{-2}. \]

c) \[ v = \frac{1}{y_1^2} e^{-\int p(x) \, dx} \int e^{\int p(x) \, dx} \, dx \]

\[ = \int \frac{1}{x^2} e^x + x + \ln x \, dx = \int \frac{1}{x^2} e^x \, dx = e^x + C. \]

So \[ v_2 = \sqrt{y_1} = e^x. \]

The general solution is then

\[ y = A e^x + B e^x. \]

8. \[ x(e^x)' - (2x + 1)(e^x)' + (x + 1)e^x = 0. \]

So \[ y_1 \] is a solution.

\[ v = \int \frac{1}{y_1^2} \, dx = \int \frac{1}{e^x} \, dx \]

\[ = \int \frac{1}{2x} e^{2x + \ln x} \, dx = \int x \, dx = \frac{x^2}{2}. \]
\[ y_2 = \sqrt{y_1} = \frac{x^2}{2} e^x. \]

The general solution is
\[ y = A e^x + B x^2 e^x. \]

§2.5.1. We treat this as a calculus problem.
Differentiate (2.22) with respect to \( w \) and set the derivative equal to 0.
\[
\frac{d}{dw} \left[ F_0 \left[ \left( k - \omega^2 M \right)^2 + \omega^2 c \right] \right]^{-\frac{1}{2}} = 0
\]

\[
F_0 \left[ \left( k - \omega^2 M \right)^2 + \omega^2 c \right]^{-\frac{3}{2}} \cdot \left[ 2 \left( k - \omega^2 M \right) (-2 \omega M) \right] + 2 \omega c = 0
\]

So it must be that
\[
2 \left( k - \omega^2 M \right) (-2 \omega M) + 2 \omega c = 0
\]
\[
-4 \omega k M + 4 \omega^3 M^2 + 2 \omega c = 0
\]
\[
-4 \omega k M + 4 \omega^2 M^2 + 2 c = 0
\]
\[
4 \omega^2 M^2 = -2 c + 4 k M
\]
\[
\omega^2 = \frac{-2 c + 4 k M}{4 M^2}
\]
\[
\omega = \sqrt{\frac{-c/2 + k M}{M}}
\]
Now you can plug this into (2, 22) to get the maximum amplitude.

Of course

\[ w = \sqrt{\frac{c^2 + \omega^2}{m}} = \sqrt{\frac{\omega^2}{m}} \]

which is the natural frequency.

2. Now

\[ x(t) = \frac{x_0 \sqrt{a^2 + b^2}}{a} e^{-bt} \cos (at - \theta) \]

We want to maximize

\[ e^{-bt} \cos (at - \theta) \]

So differentiate with respect to \( t \) and set equal to 0.

The derivative is

\[ -be^{-bt} \cos (at - \theta) - e^{-bt} \alpha \sin (at - \theta) = 0 \]

\[ -b \cos (at - \theta) - \alpha \sin (at - \theta) = 0 \]

\[ 1 + \frac{\alpha}{b} \tan (at - \theta) = 0 \]

\[ \frac{\alpha}{b} \tan (at - \theta) = -1 \]
\[
\tan (\alpha t - \theta) = -\frac{b}{a}
\]
\[
\alpha t - \theta = -\tan^{-1} \left( -\frac{b}{a} \right)
\]
\[
t = \frac{1}{\alpha} \tan^{-1} \left( -\frac{b}{a} \right) + \frac{\theta}{\alpha}
\]

But
\[
\theta = \tan^{-1} \left( \frac{b}{a} \right).
\]

So \( t = 0 \),

Thus \( \theta \) is a critical point. It follows from periodicity that \( T, 2T, \) etc., will be critical points (remember that \( T \) is)
\[
T = \frac{2\pi}{\sqrt{k - c^2/4m^2}} = \frac{2\pi}{\sqrt{2 - k/c^2}}
\]

6. We need to divide 128 lb. by the gravitational force 32 to convert to mass. So the ODE is
\[
4 \frac{d^2x}{dt^2} + 64x = 32 \sin 4t
\]
\[
\frac{d^2x}{dt^2} + 16x = 8 \sin 4t
\]

The solution to the homogeneous case is \( \cos 4t, \sin 4t \).
So our variation of parameters gives for a particular solution will be

\[ x(t) = A \cos 4t + B \sin 4t. \]

\[ \frac{d^2}{dt^2} \left( A \cos 4t + B \sin 4t \right) + 16 \left( A \cos 4t + B \sin 4t \right) = 8 \sin 4t \]

\[ (A \cdot 0 + 2A \cdot 1 \cdot (-4 \sin 4t)) - A \cdot 16 \cos 4t \]

\[ + B \cdot 0 + 2B \cdot 1 \cdot (4 \cos 4t) - B \cdot (16 \sin 4t) \]

\[ + 16 \left( A \cos 4t + B \sin 4t \right) = 8 \sin 4t \]

\[ \cos 4t (8B) + \sin 4t (-8A) + t \cos 4t (-16A + 16A) \]

\[ + \sin 4t (-16B + 16B) = 8 \sin 4t \]

\[ 8B \cos 4t - 8A \sin 4t = 8 \sin 4t \]

\[ \therefore B = 0, \quad A = -1. \]

So \( p(t) = -t \cos 4t. \)

The solution to the original ODE \((*)\) is

\[ x(t) = -t \cos 4t + A \cos 4t + B \sin 4t \]
Now using the fact,
\[ \frac{1}{2} = x(0) = A \]
\[ 0 = \frac{dx}{dt}(0) = \cos \delta t + 4t \sin \delta t - 4A \sin \delta t + \frac{1}{4} \beta \cos \delta t \]
\[ = -1 + \frac{1}{4} \beta \]
so \[ \beta = \frac{1}{4} \].

The solution to the IVP is
\[ x(t) = -t \cos \delta t + \frac{1}{2} \cos \delta t + \frac{1}{4} \sin \delta t. \]
Because of the first term, \( |x(t)| \) blows up as \( t \to \infty \).

\section{2.6}

12) As indicated in my email, the mean distance is generally agreed to be the length of the semi-major axis of the ellipse.

Since \[ \frac{\ell^2}{4} = \frac{4 \pi a^3}{26} \]
we see that
\[ \frac{4}{3} \pi r^3 = GM \frac{T^2}{2} \]
so
\[ a = \left( \frac{GM T^2}{4 \pi^2} \right)^{\frac{1}{3}}. \]

Now 88 days is 0.24 of a earth year. Thus \( T = 0.24 \). We know what \( G, M \) are from the discussion of Kepler's Third Law. So it is easy to calculate \( a \).

b) Saturn's mean distance from the sun is
\[ 1431 \times 10^6 = 1.431 \times 10^9 \text{ km}. \]

This is \( a \). We can put that value into the equation
\[ T^2 = \frac{4 \pi^2 a^3}{GM} \]
and solve for \( T \).

5. As noted in my email, \( h = 11G' \)
and \( k = GM \).
Thus it is clear that

\[ T = \frac{4\pi^2}{GM} \cdot \frac{3}{k} = \frac{4\pi^2}{k} a^3.\]

Now we know that the ellipse is

\[ r = \frac{h^2/k}{1 + e \cos \theta}.\]

The extreme values occur when \( \theta = 0, \pi \).

So \( a = \frac{1}{2} \left[ \frac{h^2/k}{1 + e} + \frac{h^2/k}{1 - e} \right] \)

\[ = \frac{h^2}{k(1 - e^2)} = \frac{h^2}{k} \frac{a^2}{b^2}.\]

\((a^2 - b^2 = c^2(1 - e^2)).\)

So \( \left( \frac{4\pi^2}{k} \right) a^3 = \frac{4\pi^2}{k} d \cdot \left( \frac{b^2 k}{h^2} \right) = \frac{4\pi^2}{k} d^2 b^2.\)
2) If the initial mean distance

time (but of each) then it's

\(a' = 2a\), where \(a\) is earth's semimajor

distance. So

\[
T'^2 = a'^3 \Rightarrow T' = 8\frac{2}{3}
\]

\[
T' = 2\sqrt[3]{2}a
\]

which is \(2\sqrt[3]{2}\) times the period of an

eartly orbit.

b) If instead it's 3 times then

\(a' = 3a\), so

\[
T'^2 = a'^3 \Rightarrow T' = 27\frac{2}{3}
\]

\[
T' = \sqrt[3]{27}a
\]

which is \(\sqrt[3]{27}\) times the period of an

earthly orbit.
c) If instead, is 25 ft, then
\[ a' = 25 \, \text{a}, \quad t' = 2^{3/2} = 15.625 \, 2^{3/2} \]
\[ \therefore t' = \sqrt{15.625} \, \text{a}. \]

**2.7**

1. \[ y''' - 3y'' + 2y' = 0 \]
   \[ r^3 - 3r^2 + 2r = 0 \]
   \[ r(r^2 - 3r + 2) = 0 \]
   \[ r(r - 1)(r - 2) = 0 \Rightarrow r = 0, 1, 2. \]
   General sol. is \( y = A_1 e^{0x} + A_2 e^x + A_3 e^{2x}. \)

2. \[ y''' - 3y'' + 4y' - 2 = 0 \]
   \[ r^3 - 3r^2 + 4r - 2 = 0. \]
   Note that \( r = 1 \) is a solution. So
   \[ (r - 1)(r^2 - 2r + 2) = 0 \]
   \[ r - 2 \pm \sqrt{4 - 8} = 2 \pm 2i \]
   \[ r = 2 \pm 2i \]
   \[ (r - 1)(r - (1 + i))(r - (1 - i)) = 0 \]
   General sol.: \( y = A e^x + B e^x \cos x + C e^x \sin x. \)
7. \( y^{(4)} - y = 0 \)
\[ r^4 - 1 = 0 \]
\[ (r^2 - 1)(r^2 + 1) = 0 \]
\[ r^2 + 1 = \pm i \]

General solution is \( y = Ae^x + Be^{-x} + C \cos x + D \sin x \).

8. \( y^{(4)} + 5y'' + 6y = 0 \)
\[ r^4 + 5r^2 + 6 = 0 \]
\[ (r^2 + 2)(r^2 + 3) = 0 \]
\[ r = \pm i \sqrt{2}, \pm i \sqrt{3} \]

General solution:

\[ y = A \cos 2x + B \sin 2x + C \cos x + D \sin x \]

18. \( y''' - y' = 1 \) (*)

Homogeneous is \( y''' - y' = 0 \)
\[ r^3 - r = 0 \]
\[ r(r^2 - 1) = 0 \Rightarrow r = 0, 1, -1 \]

General sol. of homogeneous is

\[ y = Ae^x + Be^{-x} + Ce^{-x} \]
Guess a particular solution of the form

\[ y = ax^3 + bx^2 + cx + d. \]

\[ y'' - y' = 1 \]

\[ 6a - (3a^2 + 2b + c) = 1 \]
\[ 6a - 1 = 1 \]
\[ a = 0, \quad b = 0 \]
\[ \Rightarrow y = -x. \]

So \( y_p = -x \) is a particular solution. The general solution of (**) is

\[ y = -x + Ae^{0x} + Be^x + C e^{-x}. \]

\[ y = y(0) = 0 + A + B + C \]
\[ y = y'(0) = -1 + 0 + Be^0 + Ce^{-0} \]
\[ = -1 + B - C \]
\[ y = y''(0) = Be^x + Ce^{-x} \left|_0 = B + C. \right. \]

So \( A = 0, \quad B = 9/2, \quad C = -1/2. \)

The sol. of the EVP is

\[ y = -x + \frac{9}{2}e^x - \frac{1}{2}e^{-x}. \]
19.2) Note that $x = e^z$, $z = \ln x$.

Our unknown function is $y(x)$, but it is also convenient to think of it as $\phi(z)$.

So $y(x) = \phi(z)$

\[
\frac{dy}{dx} = \frac{d\phi}{dz} \frac{dz}{dx} = \frac{1}{x} \frac{d\phi}{dz} = -z \frac{d\phi}{dz}
\]

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{1}{x} \frac{d\phi}{dz} \right) = -\frac{1}{x^2} \frac{d\phi}{dz} + \frac{1}{x^2} \frac{d^2\phi}{dz^2} \frac{dz}{dx}
\]

\[
= \frac{1}{x^2} \left[ \frac{d^2\phi}{dz^2} - \frac{d\phi}{dz} \right] = -2z \left( \frac{d^2\phi}{dz^2} - \frac{d\phi}{dz} \right)
\]

\[
\frac{d^3y}{dx^3} = -\frac{2}{x^3} \left[ \frac{d^2\phi}{dz^2} - \frac{d\phi}{dz} \right] + \frac{1}{x^2} \left[ \frac{d^3\phi}{dz^3} \frac{dz}{dx} - \frac{d^2\phi}{dz^2} \frac{d^2z}{dx^2} \right]
\]

\[
= \frac{d^3\phi}{dz^3} \frac{1}{x^3} + \frac{d^2\phi}{dz^2} \left( -\frac{2}{x^3} - \frac{1}{x^3} \right)
\]

\[
+ \frac{d\phi}{dz} \left( \frac{2}{x^3} \right)
\]

\[
= e^{-3z} \frac{d^3\phi}{dz^3} - 3e^{-3z} \frac{d^2\phi}{dz^2} + 2e^{-3z} \frac{d\phi}{dz}
\]
So our differential equation becomes:

\[ e^{-3z} \left( e^{\frac{\alpha}{r^3}} - 3 e^{\frac{\alpha}{r^2}} \frac{d\phi}{dr} + 3 e^{-\frac{\alpha}{r}} \frac{d\phi}{dr} \right) + 3 e^{-2z} \left( e^{\frac{\alpha}{r^2}} - e^{\frac{\alpha}{r}} \right) = 0 \]

\[ \frac{d^3\phi}{dr^3} - 3 \frac{d^2\phi}{dr^2} + 2 \frac{d\phi}{dr} + 3 \frac{d^2\phi}{dr^2} - 3 \frac{d\phi}{dr} = 0 \]

\[ \frac{d^3\phi}{dr^3} - \frac{d\phi}{dr} = 0 \]

\[ r^3 - r \geq 0 \]

\[ r(r-2)(r+1) = 0 \]

\[ \phi = Ae^{0z} + Be^{z} + Ce^{-z} \]

\[ \cdot \cdot \cdot \eta = A + Bx + C \frac{1}{x} \cdot \]