§ 6.2

1b) \( y = A \sin \sqrt{\lambda} x + B \cos \sqrt{\lambda} x \)

Since \( y(0) = 0 \) we find that \( B = 0 \). So
\[
y = A \sin \sqrt{\lambda} x,
\]
Since \( y(2\pi) = 0 \), we see that \( \sqrt{\lambda} \cdot 2\pi \) is a multiple of \( \pi \).
\[
\sqrt{\lambda} \cdot 2\pi = n\pi
\]
\[
\sqrt{\lambda} = n/2
\]
\[
\lambda = \left( \frac{n}{2} \right)^2.
\]
These are the eigenvalues and the corresponding eigenfunctions are
\[
y_m(x) = \sin \frac{nx}{2}.
\]

d) \( y = A \sin \sqrt{\lambda} x + B \cos \sqrt{\lambda} x \)

Since \( y(0) = 0 \) we find that \( B = 0 \). So
\[
y = A \sin \sqrt{\lambda} x.
\]
Since \( y(L) = 0 \), we see that \( \sqrt{\lambda} L \) is a multiple of \( \pi \).
So \[ \sqrt{X} L = n \pi \]
\[ \sqrt{X} = \frac{n \pi}{L} \]
\[ X = \frac{n^2 \pi^2}{L^2} \]

These are the eigenvalues and the corresponding eigenfunctions are
\[ y_n(x) = \sin \frac{n \pi x}{L} \]

2. a) \[ y_x = F'(x+at) + G'(x-at) \]
\[ y_{xx} = F''(x+at) + G''(x-at) \]
\[ y_t = a F'(x+at) - a G'(x-at) \]
\[ y_{tt} = a^2 F''(x+at) + a^2 G''(x-at) \]
Thus \[ a^2 y_{xx} = y_{tt} \]

b) We see that
\[ \frac{\partial}{\partial x} = \frac{\partial}{\partial x} \frac{\partial}{\partial x} + \frac{\partial}{\partial \beta} \frac{\partial}{\partial \beta} = \frac{\partial}{\partial x} + \frac{\partial}{\partial \beta} \]
and
\[ \frac{\partial}{\partial t} = \frac{\partial}{\partial x} \frac{\partial}{\partial x} + \frac{\partial}{\partial \beta} \frac{\partial}{\partial \beta} = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial \beta^2} \]
Thus
\[
\frac{\partial^2 y}{\partial x^2} = \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial \beta} \right) \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial \beta} \right) y
\]
\[
= \left( \frac{\partial^2}{\partial x^2} + 2 \frac{\partial^2}{\partial x \partial \beta} + \frac{\partial^2}{\partial \beta^2} \right) y
\]

Thus
\[
\frac{\partial^2 y}{\partial t^2} = (a^2 \frac{\partial^2}{\partial x^2} - a^2 \frac{\partial^2}{\partial \beta^2}) \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial \beta} \right) y
\]
\[
= a^2 \left[ \frac{\partial^2}{\partial x^2} - 2 \frac{\partial^2}{\partial x \partial \beta} + \frac{\partial^2}{\partial \beta^2} \right] y
\]

Thus
\[
0 = a^2 \frac{\partial^2 y}{\partial x^2} - a^2 \frac{\partial^2 y}{\partial t^2}
\]
\[
= \frac{a^4 y}{\partial x^2} \frac{\partial}{\partial x} \frac{\partial}{\partial \beta}
\]
\[
\frac{\partial^2 y}{\partial t \partial \beta} = 0. \quad (\star)
\]

From (\star) we see that
\[
\frac{\partial y}{\partial x} = \phi (x)
\]

for an arbitrary function \( \phi \), so
\[
y = \phi (x) + \psi (\beta)
\]

for arbitrary functions \( \phi \), \( \psi \). But this says
\[
y = \phi (x + at) + \psi (x - at).
\]
5. We know that
\[ y = \sum_{j=1}^{\infty} b_j \sin jx \cos jt, \]
So
\[ \sin x = f(x) = y(x,0) = \sum_{j=1}^{\infty} b_j \sin jx. \]
We conclude that \( b_1 = c \) and all other \( b_j = 0 \), so
\[ y = c \sin x \cos t. \]

The value of \( c \) only affects the amplitude. The "shape" of the string is independent of \( c \).
If instead \( f(x) = \sin x \sin x \)
then \( b_1 = c \) and all other \( b_j = 0 \), so
\[ y = c \sin x \cos t. \]
Again, the shape is independent of \( c \).
The factor \( c \) only affects the amplitude.
\[ \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} \]

\[ w(x, 0) = f(x), \quad w(0, t) = w_1, \quad w(\pi, t) = w_2. \]

Take \( \alpha = 1 \) for simplicity. From Example 6.2, let \( g(x) = w_1 + \frac{1}{\alpha^2} (w_2 - w_1) \). Set

\[ F(x) = f(x) - g(x). \]

We know that

\[ w(x, t) = \sum_j b_j e^{-\frac{j^2 \pi^2 t}{\alpha^2}} \sin jx. \]

We set

\[ F(x) = w(x, 0) = \sum_j b_j \sin jx. \]

and solve for the \( b_j \) as usual.

Then we let our new solution be

\[ \tilde{w}(x, t) = \sum_{j=1}^{\infty} b_j e^{-\frac{j^2 \pi^2 t}{\alpha^2}} \sin jx + g(x). \]

This solves the heat equation,

\[ \tilde{w}(0, t) = w_1, \quad \tilde{w}(\pi, t) = w_2, \quad \text{and} \]

\[ \tilde{w}(x, 0) = F(x) + g(x) = f(x). \]
3. \( \frac{\partial^2 w}{\partial x^2} = \frac{\partial w}{\partial t} + c (w - w_0) \)

We take \( w_0 = 0 \) and, for simplicity, take \( a = 1, c = 1 \). So the equation becomes

\( \frac{\partial^2 w}{\partial x^2} = \frac{\partial w}{\partial t} + w \)

We guess \( w(x, t) = \alpha(x) \beta(t) \). So

\( \alpha''(x) \beta(t) = \alpha(x) \beta'(t) + \alpha'(x) \beta(t) \)

\( \beta(t) [\alpha''(x) - \alpha'(x)] = \alpha(x) \beta'(t) \)

\( \frac{\alpha''(x) - \alpha'(x)}{\alpha(x)} = \frac{\beta'(t)}{\beta(t)} \)

\( \therefore \frac{\alpha''(x)}{\alpha(x)} - 1 = K \frac{\beta'(t)}{\beta(t)} \)

So \( \beta(t) = Ce^{Kt} \).

\( \frac{\alpha''(x)}{\alpha(x)} = 1 + K \)

Because of the boundary conditions,

\( \alpha(x) = \sin \sqrt{-(K+1)x} \)

\( K + 1 = -j^2 \) or \( K = -j^2 - 1 \)

Thus \( w = \sum_j b_j e^{-j^2 - 1} \frac{t}{\sin jx} \).
If the ends of the rod are insulated then
\[
\frac{\partial w}{\partial x}(0, t) = 0 \quad \text{and} \quad \frac{\partial w}{\partial x}(\pi, t) = 0.
\]

So the solution of the heat equation becomes
\[
w(x, t) = \sum b_j e^{-j^2 t} \cos jx.
\]

We know that
\[
f(x) = w(x, 0) = \sum b_j \cos jx
\]
so the \(b_j\) are the usual Fourier cosine coefficients.

\[\text{To Do:}\]
1. b) This \(f\) is odd so all \(b_j = 0\).

\[
b_j = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin j\theta \, d\theta = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos j\theta \, d\theta + \frac{1}{\pi} \int_{-\pi}^{\pi} \cos j\theta \, d\theta
\]

\[
= -\frac{1}{\pi} \left[ \frac{\sin j\theta}{j} \right]_{-\pi}^{\pi} - \frac{1}{\pi} \left[ \frac{\sin j\theta}{j} \right]_{-\pi}^{\pi} + \frac{1}{\pi} \sin j\theta \left|_{-\pi}^{\pi} \right| = 2(-1)^{j+1}
\]

Thus
\[
w(r, \theta) = \sum_{j=2}^{\infty} \frac{2(-1)^{j+1}}{r^j} \sin j\theta.
\]
1) \[ a_j = \frac{1}{\pi} \int_0^{\pi} 1 \cos j \theta \, d\theta = \frac{1}{\pi} \sin j \theta \bigg|_0^{\pi} = 0 \]
   for \( j \geq 1 \).

   \[ a_0 = \frac{1}{\pi} \int_0^{\pi} 1 \, d\theta = 1. \]

   \[ b_j = \frac{1}{\pi} \int_0^{\pi} 1 \sin j \theta \, d\theta = -\frac{1}{\pi} \cos j \theta \bigg|_0^{\pi} \]
   \[ = \frac{1}{\pi j} (\cos j \cdot 0 + 1) = \frac{1}{\pi j} [1 - (-1)^j] = \left\{ \begin{array}{ll} 1/\pi j & \text{if } j \geq 1 \\ 0 & \text{if } j = 0 \end{array} \right. \]

So \( w(r, \theta) = \frac{1}{2} + \sum_{l=1}^{\infty} \frac{1}{\pi l} r \sin 2lx \)

2. Let \( \Phi : D(0, R) \to D(0, 1) \) be given by \( (x, y) \mapsto (x/R, y/R) \).

   If \( f(\theta) \) is a boundary function on \( \partial D(0, R) \),
   then \( f \) will have a Fourier expansion
   \[ f(\theta) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j \theta + \sum_{j=1}^{\infty} b_j \sin j \theta, \]
   Then \( w(r, \theta) = \frac{a_0}{2} + \sum_{j=2}^{\infty} a_j r \cos j \theta + \sum_{j=2}^{\infty} b_j r \sin j \theta \).
given a solution to the Dirichlet problem on $D(0, R)$ and 
\[ w_0 = \frac{2}{2} + \sum_{j=1}^{\infty} \left( \frac{m}{R} \right) e^{-2\pi j R} + \sum_{j=1}^{\infty} \frac{c_j}{j} \sin(j \pi r) \]
gives a solution on $D(0, R)$.

\[ \frac{\partial w}{\partial r} = \cos \theta \frac{\partial w}{\partial x} + \sin \theta \frac{\partial w}{\partial y} \]
\[ \frac{\partial w}{\partial \theta} = -r \sin \theta \frac{\partial w}{\partial x} + r \cos \theta \frac{\partial w}{\partial y} \]

Multiply the first equation by $r \sin \theta$ and the second by $\cos \theta$ and add. We get 
\[ \frac{\partial w}{\partial \theta} = r \sin \theta \frac{\partial w}{\partial x} + \cos \theta \frac{\partial w}{\partial y} \]
\[ \frac{\partial}{\partial y} = \sin \theta \frac{\partial w}{\partial r} + \frac{1}{r \cos \theta} \frac{\partial w}{\partial \theta} \]

Now multiply the first equation by $r \cos \theta$ and the second by $\sin \theta$ and subtract. We get 
\[ \frac{\partial w}{\partial x} = r \cos \theta \frac{\partial w}{\partial r} - \sin \theta \frac{\partial w}{\partial \theta} \]
\[ \frac{\partial}{\partial x} = \cos \theta \frac{\partial w}{\partial r} - \frac{1}{r \sin \theta} \frac{\partial w}{\partial \theta} \]
Now

\[ A_w = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) w \]

\[ = (\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta \partial}{r}) (\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}) w \]

\[ + (\sin \theta \frac{\partial}{\partial r} + \cos \theta \frac{\partial}{\partial \theta}) (\sin \theta \frac{\partial}{\partial r} + \cos \theta \frac{\partial}{\partial \theta}) w \]

\[ = \cos^2 \theta \frac{\partial^2 w}{\partial r^2} + \cos \theta \sin \theta \frac{\partial^2 w}{\partial r \partial \theta} - \cos \theta \sin \theta \frac{\partial^2 w}{\partial \theta \partial r} - \sin^2 \theta \frac{\partial^2 w}{\partial \theta^2} + \frac{\sin \theta}{r} \frac{\partial w}{\partial r} r \frac{\partial w}{\partial \theta} \]

\[ + \sin^2 \theta \frac{\partial^2 w}{\partial r^2} + \sin \theta \cos \theta \frac{\partial^2 w}{\partial r \partial \theta} + \sin \theta \cos \theta \frac{\partial^2 w}{\partial \theta \partial r} - \sin \theta \cos \theta \frac{\partial^2 w}{\partial \theta^2} \]

\[ + \cos^2 \theta \frac{\partial^2 w}{\partial r^2} + \cos \theta \sin \theta \frac{\partial^2 w}{\partial r \partial \theta} + \cos \theta \sin \theta \frac{\partial^2 w}{\partial \theta \partial r} - \cos \theta \sin \theta \frac{\partial^2 w}{\partial \theta^2} \]

\[ - \cos \sin \theta \frac{\partial^2 w}{\partial r \partial \theta} + \cos^2 \theta \frac{\partial^2 w}{\partial \theta^2} \]

\[ + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r} \frac{\partial w}{\partial r} \]

\[ = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r} \frac{\partial w}{\partial r} \]