On Limits of Sequences of Holomorphic Functions

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Abstract: *We study functions which are the pointwise limit of a sequence of holomorphic functions. In one complex variable this is a classical topic, though we offer some new points of view and new results. Some novel results for solutions of elliptic equations will be treated. In several complex variables the question seems to be new, and we explore some new results.*

0 Introduction

It is a standard and well known fact from complex function theory (which appears to be due to Weierstrass (see [WEI]); closely related results appear in [STE] and [VIT]) that if $\{f_j\}$ is a sequence of holomorphic functions on a planar domain Ω and if the sequence converges *uniformly on compact subsets of* Ω then the limit function is holomorphic on Ω. Certainly this result is one of several justifications for equipping the space of holomorphic functions on Ω with the compact-open topology (see also [LUR], where this point of view is developed in detail from the perspective of functional analysis).

Considerably less well known is the following result of William Fogg Osgood [OSG]:

Theorem 1 *Let $\{f_j\}$ be a sequence of holomorphic functions on a planar domain Ω. Assume that the f_j converge* pointwise *to a limit function f on Ω. Then f is holomorphic on a dense, open subset of Ω. The convergence is uniform on compact subsets of the dense, open set.*

This result is not completely obvious; it is certainly surprising and interesting. For completeness, we now offer a proof of the theorem:

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Proof of the Theorem: Let U be a nonempty open subset of Ω with compact closure in Ω. Define, for $k = 1, 2, \ldots$,

$$S_k = \{z \in \overline{U} : |f_j(z)| \leq k \text{ for all } j \in \mathbb{N}\}.$$

Since the f_j converge at each $z \in \overline{U}$, certainly the set $\{f_j(z) : j \in \mathbb{N}\}$ is bounded for each fixed z. So each $z \in \overline{U}$ lies in some S_k. In other words,

$$\overline{U} = \bigcup_k S_k .$$

Now of course $\overline{U}$ is a complete metric space (in the ordinary Euclidean metric), so the Baire category theorem tells us that some S_k must be "somewhere dense" in $\overline{U}$. This means that $\overline{S_k}$ will contain a nontrivial Euclidean metric ball (or disc) in $\overline{U}$. Call the ball $\mathcal{B}$. Now it is a simple matter to apply Montel's theorem on $\mathcal{B}$ to find a subsequence f_{j_k} that converges uniformly on compact sets to a limit function g. But of course g must coincide with f, and g (hence f) must be holomorphic on $\mathcal{B}$.

Since the choice of U in the above arguments was arbitrary, the conclusion of the theorem follows. □

Remark: An alternative approach, which avoids the explicit use of Montel's theorem, is as follows. Once one has identified an S_k whose closure contains a ball or disc $D(P, r)$, let γ be a simple closed curve in $D(P, r)$. Then of course the image $\widetilde{\gamma}$ of γ is a compact set. Let $\epsilon > 0$. By Lusin's theorem, the sequence f_j converges uniformly on some subset $E \subseteq \widetilde{\gamma}$ with the property that the linear measure of $\widetilde{\gamma} \setminus E$ is less than ϵ. Let K be a compact subset of the open region surrounded by γ, and let $\delta > 0$ be the Euclidean distance of K to $\widetilde{\gamma}$. Let $\epsilon^* > 0$ and choose $J > 0$ so large that when $j, k > J$ then

$$|f_j(z) - f_k(z)| < \epsilon^*$$

for all $z \in E$. Then, for $w \in K$,

$$\begin{aligned}
|f_j(w) - f_k(w)| &= \left| \frac{1}{2\pi i} \oint_\gamma \frac{f_j(\zeta) - f_k(\zeta)}{\zeta - w} \, d\zeta \right| \\
&\leq \frac{1}{2\pi} \int_E \frac{\epsilon^*}{\delta} \, ds + \frac{1}{2\pi} \int_{\widetilde{\gamma} \setminus E} \frac{2k}{\delta} \, ds \\
&\leq \frac{\epsilon^* \cdot |E|}{2\pi \cdot \delta} + \frac{\epsilon \cdot 2k}{2\pi \cdot \delta} .
\end{aligned}$$

Thus we see that we have uniform convergence on K. And the holomorphicity follows as usual. $\square$

The next example is inspired by ideas in [ZAL, pp. 131–133]. It demonstrates that Osgood's theorem has substance, and describes a situation that actually occurs. A thorough discussion of many of the ideas treated here—from a somewhat different point of view—appears in [BEM]. In fact [BEM] presents quite a different contruction of an example that illustrates Theorem 1.

EXAMPLE 1 Let

$$U = \{z \in \mathbb{C} : |\mathrm{Re}\, z| < 1, |\mathrm{Im}\, z| < 1\}\,.$$

For $j = 1, 2, \ldots$, define

$$S_j = \{z \in U : \mathrm{Re}\, z = 0 \text{ or } \mathrm{Im}\, z = 0, |\mathrm{Re}\, z| \le 1{-}1/[j{+}2]\;,\; |\mathrm{Im}\, z| \le 1{-}1/[j{+}2]\}\,.$$

Also define

$$T_j = \{z \in U : 1/[j{+}2] \le |\mathrm{Re}\, z| \le 1{-}1/[j{+}2], 1/[j{+}2] \le |\mathrm{Im}\, z| \le 1{-}1/[j{+}2]\}\,.$$

We invite the reader to examine Figure 1 to appreciate these sets.

Now, for each j, we apply Runge's theoreom. Notice that for each j the complement of $S_j \cup T_j$ is connected, so that we can push the poles of the approximating functions to the complement of U. We are able then to produce for each j a holomorphic function f_j on U such that

$$|f_j(z) - 0| < \frac{1}{j} \quad \text{for} \quad z \in T_j\,,$$

$$|f_j(z) - 1| < \frac{1}{j} \quad \text{for} \quad z \in S_j\,.$$

Then it is easy to see that the sequence $\{f_j\}$ converges pointwise to the function f given by

$$f(z) = \begin{cases} 0 & \text{if} \quad z \in U \setminus \{z \in U : \mathrm{Re}\, z = 0 \text{ or } \mathrm{Im}\, z = 0\} \\ 1 & \text{if} \quad z \in U \cap \{z \in U : \mathrm{Re}\, z = 0 \text{ or } \mathrm{Im}\, z = 0\}\,. \end{cases}$$

Thus the limit function f is holomorphic on a dense open subset of U, and the exceptional set is the two axes in U. $\square$

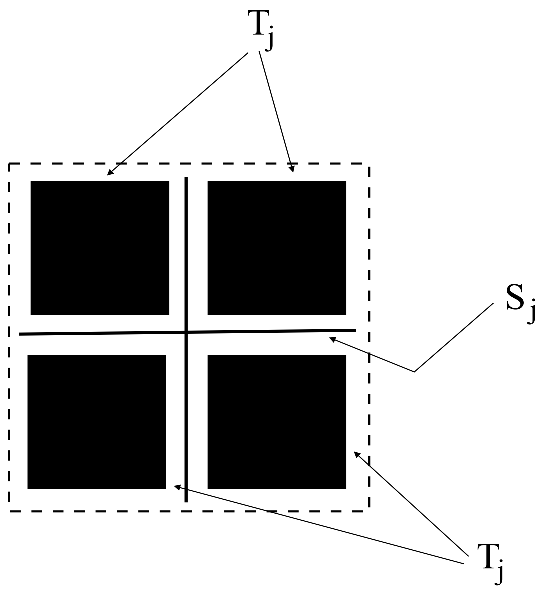

Figure 1: The sets S_j and T_j.

One might ask what more can be said about the open, dense set V on which the limit function f is holomorphic. Put in other words, what can one say about $\Omega \setminus V$? Lavrentiev [LAV] was the first to give a characterization of those open sets on which a pointwise convergent sequence of holomorphic functions can converge. Siciak [SIC] has given a rather different answer in the language of capacity theory.

In fact a suitable version of the theorem is true for harmonic functions, or more generally for solutions of a uniformly elliptic partial differential equation of second order. We shall prove such results later in the present paper.

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1 More Results on Planar Domains

Our first new result for planar domains concerns harmonic functions:

Theorem 2 *Let $\{f_j\}$ be a sequence of harmonic functions on a planar domain Ω. Assume that the f_j converge pointwise to a limit function f on Ω.*

Then f is harmonic on a dense open subset of Ω.

Sketch of the Proof of the Theorem: Proceed as in the proof of the result for holomorphic functions. It is certainly true that a collection of harmonic functions on a planar domain that is uniformly bounded on compacta will have a subsequence that converges uniformly on compact sets. This follows from easy estimates on the Poisson kernel. The rest of the argument is the same as before. □

Theorem 3 *Let $\mathcal{L}$ be a uniformly elliptic operator of order 2 on a planar domain Ω. Let $\{f_j\}$ be a sequence of functions that are annihilated by $\mathcal{L}$ on Ω. Assume that the f_j converge pointwise to a limit function f on Ω. Then f is annihilated by $\mathcal{L}$ on a dense open subset of Ω.*

Proof: The proof is the same as the last result. The only thing to check is that a collection of functions annihilated by $\mathcal{L}$ that is bounded on compact sets will have a subsequence that converges uniformly on compact sets. This will follow, as in the harmonic case, from the Poisson formula for $\mathcal{L}$. The rest of the argument is the same. □

Theorem 4 *Let $\{f_j\}$ be a sequence of holomorphic functions on a planar domain Ω. Suppose that there is a constant $M > 0$ such that $|f_j(z)| \leq M$ for all j and for all $z \in \Omega$. Assume that the f_j converge pointwise to a limit function f on Ω. Then f is holomorphic on all of Ω.*

Remark: Of course the new feature in this last theorem of Stieltjes [STE] is that we are assuming that the family $\{f_j\}$ is uniformly bounded. This Tauberian hypothesis gives a stronger conclusion. The proof will now be a bit different.

Proof: Let U be an open subset of Ω. Then the argument from the proof of Theorem 1 applies immediately on U. Thus the limit function is holomorphic on U. Since the choice of U was arbitrary, we are finished. □

In fact there is a much weaker condition (than in the last theorem) that will give the same result:

Theorem 5 *Let $\{f_j\}$ be a sequence of holomorphic functions on a planar domain Ω. Suppose that there is a nonnegative, integrable function g on Ω such that $|f_j(z)| \leq g(z)$ for all j and for all $z \in \Omega$. Assume that the f_j converge pointwise to a limit function f on Ω. Then f may be corrected on a set of measure zero so that it is holomorphic on all of Ω.*

Proof: The proof is simplicity itself. Let φ be a C_c^∞ function on Ω. Then

$$0 = \int \frac{\partial \varphi}{\partial \bar{\zeta}}(\zeta) f_j(\zeta)\, dA(\zeta)$$

for each j. Here dA is Lebesgue area measure on $\mathbb{C}$. Now the Lebesgue dominated convergence theorem allows us to let $j \to \infty$ and infer that

$$0 = \int \frac{\partial \varphi}{\partial \bar{\zeta}}(\zeta) f(\zeta)\, dA(\zeta)\,.$$

Thus f is a weakly holomorphic function. But Weyl's lemma then tells us that f may be corrected on a set of measure zero so that it is holomorphic in the classical sense. □

The following result is discussed but not proved in [DAV]:

Theorem 6 *Let $\{f_j\}$ be a sequence of univalent holomorphic functions on a domain Ω that converges pointwise. Then the limit function is holomorphic on all of Ω.*

For the proof, one first observes that each f_j omits an arc from its image (this is a standard fact about univalent functions that follows simply from the topology of the Riemann sphere and the open mapping principle). Then one postcomposes each f_j with an automorphism ρ_j of the Riemann sphere to arrange that there are two complex values p and q that are omitted by all the f_j. [Of necessity, these automorphism will converge then to the identity.] Then Montel's classic theorem (see [MON]) yields that the family $\{\rho_j \circ f_j\}$ is normal. Unraveling the logic, we see that the original family $\{f_j\}$ is normal. The result follows immediately.

2 Results in Several Complex Variables

The first result in $\mathbb{C}^n$ is as follows.

Theorem 7 *Let $\{f_j\}$ be a sequence of holomorphic functions on a domain $\Omega \subseteq \mathbb{C}^n$. Assume that the f_j converge pointwise to a limit function f on Ω. Then f is holomorphic on a dense open subset of Ω. Also the convergence is uniform on compact subsets of the dense open set.*

Proof: The argument is the same as that for Theorem 1. We need only note that Montel's theorem is still valid. The rest of the argument is the same. $\square$

Remark: Just as in the Remark following the proof of Theorem 1, we could use the Henkin-Ramirez integral formula on small balls (see [KRA, Ch. 8]) to give an alternative proof of this result. $\square$

Theorem 8 *Let $\{f_j\}$ be a sequence of holomorphic functions on a domain $\Omega \subseteq \mathbb{C}^n$. Assume that the f_j converge pointwise to a limit function f on Ω. Let ℓ be any complex line in $\mathbb{C}^n$. Then the limit function f is holomorphic on a dense open subset of $\ell \cap \Omega$.*

Proof: Of course we simply apply the argument from the proof of Theorem 1 on $\ell \cap \Omega$. $\square$

Remark: This is a stronger result than Theorem 6. One may note that something similar could be proved with "'complex line" replaced by "complex analytic variety". It is not clear what the optimal result might be. $\square$

3 Concluding Remarks

It is clear that there is more to learn in the several complex variable setting. We would like a result that has a chance of being sharp, so that the exceptional set for convergence can be characterized (as in [SIC] for one complex variable). This matter will be explored in future papers.

References

[BEM] A. Beardon and D. Minda, On the pointwise limit of cmoplex analytic functions, *Am. Math. Monthly* 110(2003), 289–297.

[DAV] K. R. Davidson, Pointwise limits of analytic functions, *Am. Math. Monthly* 90(1984), 391–394.

[GRK] R. E. Greene and S. G. Krantz, *Function Theory of One Complex Variable*, 3rd ed., American Mathematical Society, Providence, RI, 2006.

[KRA] S. G. Krantz, *Function Theory of Several Complex Variables*, 2nd ed., American Mathematical Society, Providence, RI, 2002.

[LAV] M. Lavrentiev, Sur les fonctions d'une variable complexe représentables par des séries de polynomes, *Actualités Scientifiques et Industrielles* 441(1936), Hermannn & Cie, Paris.

[LUR] D. Luecking and L. Rubel, *Complex Analysis: A Functional Analysis Approach*, Springer-Verlag, New York, 1984.

[MON] P. Montel, *Leçons sur les Families Normales de fonctions analytiques et leur applications*, Gauthier-Villars, Paris, 1927.

[OSG] W. F. Osgood, Note on the functions defined by infinite series whose terms are analytic functions of a complex variable, with corresponding results for definite integrals, *Ann. Math.* 3(1901), 25–34.

[SIC] J. Siciak, On series of homogeneous polynomials and their partial sums, *Annales Polonici Math.* **51**(1990), 289–302.

[STE] T.-J. Stieltjes, Recherches sur les fractions continues, *Ann. Fac. Sci. Toulouse* VIIIJ(1894), 1–122.

[TIT] E. C. Titchmarsh, *The Theory of Functions*, Oxford University Press, New York, 1960.

[VIT] G. Vitali, Sopra le serie di funzioni analitiche, *Rend. Ist. Lombardo* XXXVI(1903), 772–774.

[**WEI**] K. Weierstrass, *Mathematische Werke*, Hildesheim, G. Olms, New York, Johnson Reprint, 1967.

[**ZAL**] L. Zalcman, Real proofs of complex theorems (and vice versa), *Am. Math. Monthly* 81(1974), 115–137.

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