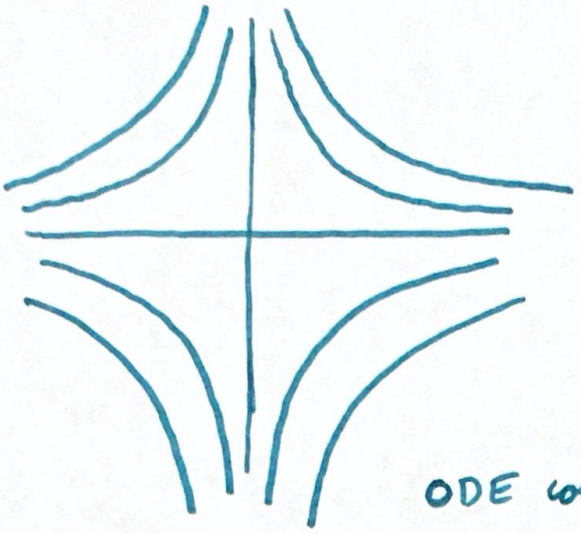
1.7.1a

$xy = c$



differentiate w.r.t. x

ODE corresponding to $xy=c$

$$0 = \frac{d}{dx}(xy) = y + \frac{dy}{dx}x \quad \rightsquigarrow \quad \frac{dy}{dx} = -\frac{y}{x}$$

ODE corresponding to orthogonal trajectories:

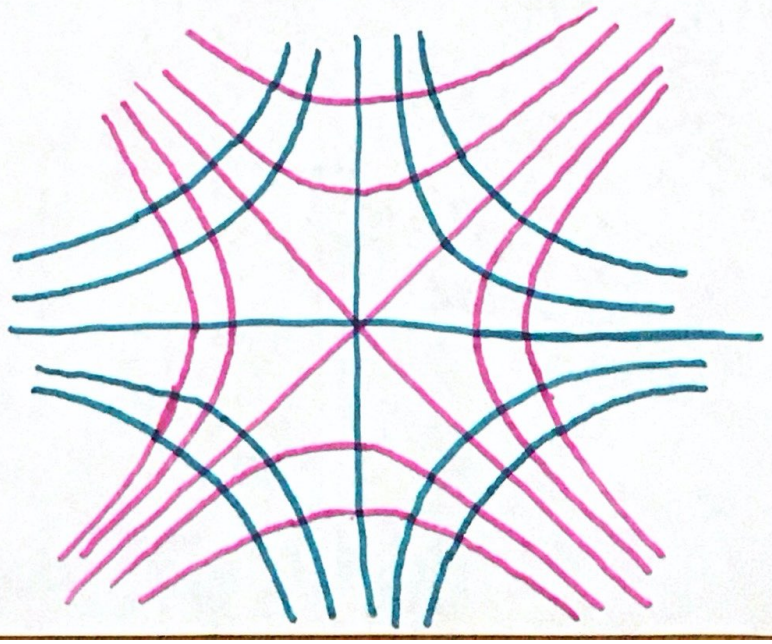
$$\frac{dy}{dx} = \frac{x}{y}$$

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

$$\boxed{y^2 - x^2 = D}$$

this is the family of hyperbolas with asymptotes $y = \pm x$.



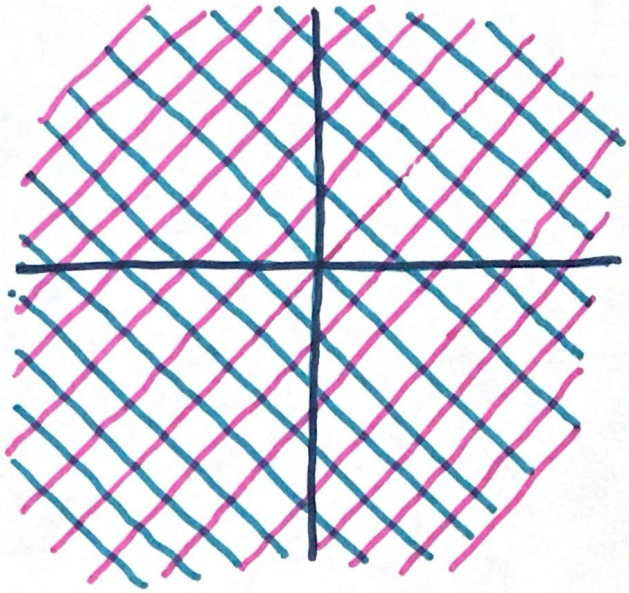
1.7.1c

$x + y = c$

$\frac{d}{dx}$ (

$1 + \frac{dy}{dx} = 0$

$\frac{dy}{dx} = -1$



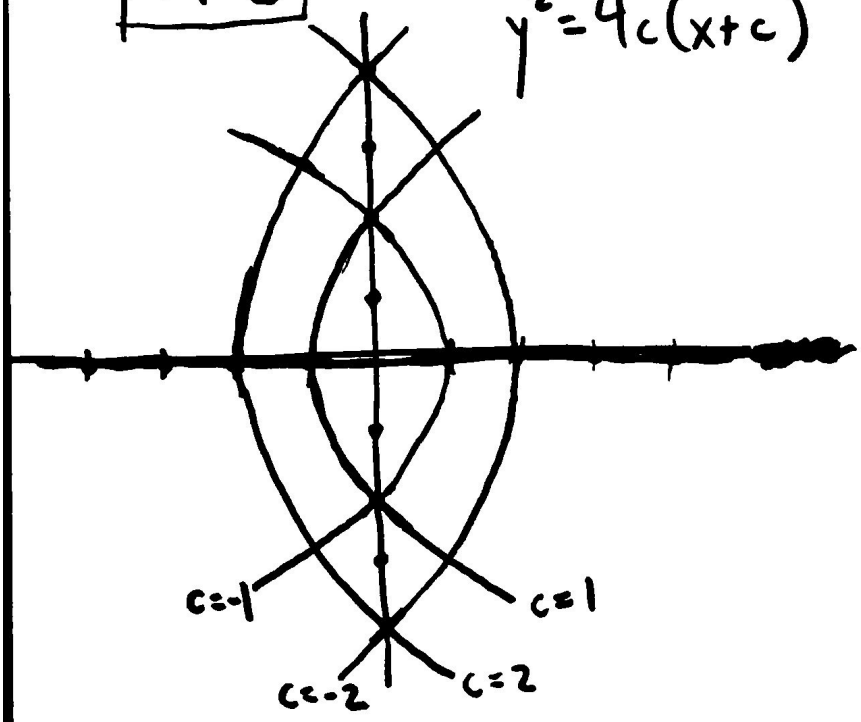
so orthogonal trajectories must solve $\frac{dy}{dx} = 1$

$\int dy = \int dx$

$y = x + c$

1.7.3

$y^2 = 4c(x+c)$



find the ODE: $2y\frac{dy}{dx} = 4c \longrightarrow \frac{dy}{dx} = \frac{2c}{y}$

we know that $y^2 = 4c(x+c) \Rightarrow 4c^2 + (4x)c - y^2 = 0$

solve for c: $c = \frac{-4x \pm \sqrt{16x^2 - 4(4)(-y^2)}}{2(4)}$

$= \frac{-x \pm \sqrt{x^2+y^2}}{2}$

so ODE is $\frac{dy}{dx} = \frac{-x \pm \sqrt{x^2+y^2}}{y}$

[redacted]

CONTINUED...

diff eq. of orthogonal trajectories is therefore

$$\frac{dy}{dx} = \frac{-y}{-x \pm \sqrt{x^2+y^2}} \cdot \frac{(-x \mp \sqrt{x^2+y^2})}{(-x \mp \sqrt{x^2+y^2})}$$

$$= \frac{xy \pm y\sqrt{x^2+y^2}}{x^2 - (x^2+y^2)}$$

$$= \frac{xy \pm y\sqrt{x^2+y^2}}{-y^2}$$

$$= \frac{-x \pm \sqrt{x^2+y^2}}{y}$$, the same equation defining our original family.

We conclude that whenever 2 curves in this family intersect, they do so orthogonally.

1.7.6 a

$y = x(\sin(x+c)) \longrightarrow \sin(x+c) = \frac{y}{x} \rightarrow c = \arcsin\left(\frac{y}{x}\right) - x$

$\frac{dy}{dx} = \sin(x+c) + x\cos(x+c)$

$\frac{dy}{dx} = \frac{y}{x} + x\cos\left(\arcsin\left(\frac{y}{x}\right)\right)$

1.7.6 c

The family of circles is defined by

$(x-c)^2 + (y-c)^2 = c^2$

we can write $c^2 = x^2 - 2xc + c^2 + y^2 - 2yc + c^2$

$\longrightarrow 0 = c^2 - (2x+2y)c + (x^2+y^2)$,

so $c = \frac{2x+2y \pm \sqrt{(2x+2y)^2 - 4(x^2+y^2)}}{2} = \frac{2x+2y \pm \sqrt{4xy}}{2}$

$\Rightarrow c = x+y \pm \sqrt{xy}$

the ODE is $2(x-c) + 2(y-c)\frac{dy}{dx} = 0$ with $c = x+y \pm \sqrt{xy}$

1.8.1 b

$$x y' - 3xy - 2y^2 = 0$$

$$x \frac{dy}{dx} = 3xy + 2y^2$$

$$\underbrace{(3xy + 2y^2)}_{\text{homogeneous of degree 2.}} dx - \underbrace{x}_{\text{homogeneous of degree 1.}} dy$$

since the degrees of homogeneity are different, we can't apply the method.

1.8.1 d

$$\underbrace{\left[x \sin\left(\frac{y}{x}\right)\right]}\frac{dy}{dx} = \underbrace{y \sin\left(\frac{y}{x}\right) + x}$$

both homogeneous of degree 1

$$\frac{dy}{dx}\sin\left(\frac{y}{x}\right) = \frac{y}{x}\sin\left(\frac{y}{x}\right) + 1$$

$$z = \frac{y}{x} \longrightarrow \frac{dy}{dx} = x\frac{dz}{dx} + z$$

$$\left(x\frac{dz}{dx} + z\right)\sin(z) = z\sin z + 1$$

$$x\frac{dz}{dx} + z = z + \frac{1}{\sin z}$$

$$x\frac{dz}{dx} = \frac{1}{\sin z} \longrightarrow \sin(z)\,dz = \frac{1}{x}\,dx$$

$$\longrightarrow -\cos(z) = \ln|x| + C$$

$$\underline{\cos\left(\frac{y}{x}\right) = \ln\frac{1}{|x|} + D}$$

1.8.3a

We know $ae \neq bd$ and we want to find h, k such that

$$\frac{dw}{dz} = \frac{dy}{dx} = F\left(\frac{a(z-h)+b(w-k)+c}{d(z-h)+e(w-k)+f}\right) = F\left(\frac{az+bw}{dz+ew}\right).$$

we see this will work iff h, k satisfy

$ah+bk=c$ and $dh+ek=f$.

We have $ah = c-bk$ and $a(dh+ek) = af$

$d(ah)+aek = af$,

so $d(c-bk)+aek = af$

$dc - bdk + aek = af$

$(ae-bd)k = af-dc$. note $ae-bd \neq 0$, so

$$\boxed{k = \frac{af-cd}{ae-bd}}.$$

similarly, $bk = c-ah$, and $bf = b(dh+ek) = bdh+ebk$

$= bdh+e(c-ah) = (bd-ae)h+ec$, so $\boxed{h = \frac{ce-bf}{ae-bd}}$.

(Note that we avoided dividing by anything we didn't know was nonzero.)

1.8.4a $\quad \frac{dy}{dx} = \frac{x+y+4}{x-y-6}$

Taking a hint from the previous problem, we would like to find h, k satisfying

$$h+k=4$$
$$h-k=-6$$

We can use the formula from that problem, or directly solve and see $h=-1$, $k=5$.

So let $x = z+1$, $y = w-5$. Then

$$\frac{dw}{dz} = \frac{z+1+w-5+4}{z+1-w+5-6} = \frac{z+w}{z-w} = \frac{1+\frac{w}{z}}{1-\frac{w}{z}}.$$

Now we make the substitution $\alpha = \frac{w}{z}$, $\frac{dw}{dz} = z\frac{d\alpha}{dz} + \alpha$. This gives

$$z\frac{d\alpha}{dz} + \alpha = \frac{1+\alpha}{1-\alpha}$$

$$z\frac{d\alpha}{dz} = \frac{1+\alpha}{1-\alpha} - \frac{\alpha(1-\alpha)}{1-\alpha} = \frac{1+\alpha^2}{1-\alpha}$$

continued...

$$\frac{1-\alpha}{1+\alpha^2}\,d\alpha = \frac{1}{z}\,dz$$

$$\int \frac{1}{1+\alpha^2}\,d\alpha - \int \frac{\alpha}{1+\alpha^2}\,d\alpha = \int \frac{1}{z}\,dz$$

$$\arctan(\alpha) - \frac{1}{2}\ln(1+\alpha^2) = \ln|z| + C$$

$$\arctan\left(\frac{w}{z}\right) - \frac{1}{2}\ln\left(1+\left(\frac{w}{z}\right)^2\right) = \ln|z| + C$$

$$\arctan\left(\frac{y+5}{x-1}\right) - \frac{1}{2}\ln\left(1+\left(\frac{y+5}{x-1}\right)^2\right) = \ln|x-1| + C$$

$$\boxed{\arctan\left(\frac{y+5}{x-1}\right) - \ln\left(\frac{|x-1|}{\sqrt{1+\left(\frac{y+5}{x-1}\right)^2}}\right) = C}$$

1.9.1b

$(xy-1)\,dx + (x^2-xy)\,dy = 0$

Then we have $\dfrac{M_y - N_x}{N} = \dfrac{x-(2x-y)}{x^2-xy} = -\dfrac{1}{x}$.

Let $\mu(x) = e^{\int -\frac{1}{x}} = e^{-\ln x} = \dfrac{1}{x}$.

$\dfrac{1}{x}(xy-1)\,dx + \dfrac{1}{x}(x^2-xy)\,dy = 0$

$\left(y - \dfrac{1}{x}\right)dx + (x-y)\,dy = 0$, exact.

∂_y $\quad$ ∂_x

$=1$ $\quad$ $=1$

so want F with $F_x = y - \frac{1}{x}$, $F_y = x - y$

$F_x = y - \frac{1}{x}$, so $F = yx - \ln|x| + g(y)$, so $F_y = x + g'(y)$.

[redacted]

So $x + g'(y) = x - y \longrightarrow g'(y) = -y \longrightarrow g(y) = -\frac{y^2}{2} + C$

Hence $F = yx - \ln|x| - y^2/2 + C$,

and $yx - \ln|x| - \frac{y^2}{2} = D$ is an implicit solution.

1.9.1e

$$(x+2)\sin y\,dx + x\cos y\,dy = 0$$

compute $\dfrac{M_y - N_x}{N} = \dfrac{(x+2)\cos y - \cos y}{x\cos y} = \dfrac{x+1}{x} = 1 + \dfrac{1}{x}$

try integrating factor $\mu(x) = e^{\int 1 + \frac{1}{x}\,dx} = xe^x$

$$\rightsquigarrow xe^x(x+2)\sin y\,dx + x^2e^x\cos y\,dy = 0$$

...you can check this is exact.

1.9.3

Let $z = x+y$. Suppose there exists $\mu(z)$ such that

$$\mu(z)M(x,y)\,dx + \mu(z)N(x,y) = 0$$

is exact. Then

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N).$$

We have

$$\frac{\partial}{\partial y}\mu M = \frac{\partial \mu}{\partial y}M + \mu\frac{\partial M}{\partial y} = \frac{d\mu}{dz}\frac{\partial z}{\partial y}M + \mu\frac{\partial M}{\partial y}$$

$$= \frac{d\mu}{dz}M + \mu\frac{\partial M}{\partial y}.$$

Similarly,

$$\frac{\partial}{\partial x}(\mu N) = \frac{d\mu}{dz}N + \mu\frac{\partial N}{\partial x}.$$

Hence

$$\frac{d\mu}{dz}M + \mu\frac{\partial M}{\partial y} = \frac{d\mu}{dz}N + \mu\frac{\partial N}{\partial x}, \text{ and...}$$

continued...

... $\frac{1}{\mu}\frac{d\mu}{dz} = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M-N}$.

Therefore $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M-N}$ is a function of z.

Conversely, suppose [illegible]

that $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M-N}$ is a function of z. Call it $g(z)$. Then the calculations above show that the solution $\mu(z)$ to $\frac{1}{\mu}\frac{d\mu}{dz} = g(z)$ will be an integrating factor.

So there is an integrating factor $\mu(z)$ iff $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M-N}$ is a function of z.