MATH 217 Solutions to Crowd Mark 5

Section 4.3

1 a) $y'' + xy' + y = 0$

$p(x) = x$ is real analytic at 0

$q(x) = 1$ " " " " "

So 0 is an ordinary point.

Guess

$$y = \sum_{j=0}^{\infty} a_j x^j$$

$$y' = \sum_{j=1}^{\infty} j a_j x^{j-1}$$

$$y'' = \sum_{j=2}^{\infty} j(j-1) a_j x^{j-2}$$

$$\sum_{j=2}^{\infty} j(j-1) a_j x^{j-2} + x \sum_{j=1}^{\infty} j a_j x^{j-1} + \sum_{j=0}^{\infty} a_j x^j = 0$$

$$\sum_{j=0}^{\infty} (j+2)(j+1) a_{j+2} x^j + \sum_{j=1}^{\infty} j a_j x^j + \sum_{j=0}^{\infty} a_j x^j = 0$$

$$2 \cdot 1 \cdot a_2 x^0 + \sum_{j=1}^{\infty} \left[(j+2)(j+1) a_{j+2} + j a_j + a_j\right] x^j + a_0 x^0 = 0$$

$$\sum_{j=1}^{\infty} \left[(j+2)(j+1) a_{j+2} + (j+1) a_j\right] x^j = (-2a_2 - a_0) x^0$$

$$-2a_2 - a_0 = 0$$

$$a_{j+2} = -\frac{a_j}{j+2}, \quad j \geq 1$$

$$a_0 = C$$

$$a_1 = D$$

$$a_2 = -\frac{a_0}{2} = -\frac{C}{2}$$

$$a_3 = -\frac{a_1}{3} = -\frac{D}{3}$$

$$a_4 = -\frac{a_2}{4} = \frac{C}{4 \cdot 2}$$

$$a_5 = -\frac{a_3}{5} = \frac{D}{5 \cdot 3}$$

So solution is

$$y = C\left[1 - \frac{x^2}{2} + \frac{x^4}{4 \cdot 2} - \frac{x^6}{6 \cdot 4 \cdot 2} + - \cdots\right]$$
$$+ D\left[x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 3} - \frac{x^7}{7 \cdot 5 \cdot 3} + - \cdots\right].$$

c) $y'' + 2xy' - y = x$

$p(x) = 2x$ is real analytic at 0

$q(x) = -1$ is real analytic at 0

So 0 is an ordinary point.

Guess

$$y = \sum_{j=0}^{\infty} a_j x^j$$

$$y' = \sum_{j=1}^{\infty} j a_j x^{j-1}$$

$$y'' = \sum_{j=2}^{\infty} j(j-1) a_j x^{j-2}$$

$$\sum_{j=2}^{\infty} j(j-1) a_j x^{j-2} + 2x \sum_{j=1}^{\infty} j a_j x^{j-1} - \sum_{j=0}^{\infty} a_j x^j = x$$

$$\sum_{j=0}^{\infty} (j+2)(j+1) a_{j+2} x^j + \sum_{j=1}^{\infty} 2j a_j x^j - \sum_{j=0}^{\infty} a_j x^j = x$$

$$2 \cdot 1 \cdot a_2 x^0 + \sum_{j=1}^{\infty} \left[(j+2)(j+1) a_{j+2} + 2j a_j - a_j \right] x^j - a_0 x^0 = x$$

$$\sum_{j=1}^{\infty} \left[(j+2)(j+1) a_{j+2} + (2j-1) a_j \right] x^j = (a_0 - 2a_2) x^0 + x$$

$$a_0 - 2a_2 = 0$$

$$3 \cdot 2 \cdot a_3 + 1 \cdot a_1 = 1$$

$$(j+2)(j+1) a_{j+2} + (2j-1) a_j = 0, \quad j \geq 2$$

$$a_{j+2} = -\frac{(2j-1) a_j}{(j+2)(j+1)}, \quad j \geq 2$$

$$a_0 = C$$

$$a_1 = D$$

$$a_2 = \frac{a_0}{2} = \frac{C}{2}$$

$$a_3 = -\frac{a_1}{3\cdot 2} = -\frac{D}{3\cdot 2}$$

$$a_4 = -\frac{3a_2}{4\cdot 3} = -\frac{C}{4\cdot 2}$$

$$a_5 = \frac{-5a_3}{5\cdot 4} = \frac{D}{4\cdot 3\cdot 2}$$

$$a_6 = \frac{-7a_4}{6\cdot 5} = \frac{7C}{6\cdot 5\cdot 4\cdot 2}$$

$$a_7 = -\frac{9a_5}{7\cdot 6} = \frac{-9D}{7\cdot 6\cdot 4\cdot 3\cdot 2}$$

$$a_8 = -\frac{11a_6}{8\cdot 7} = -\frac{11\cdot 7C}{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 2}$$

$$a_9 = -\frac{13a_7}{9\cdot 8} = \frac{13\cdot 9D}{9\cdot 8\cdot 7\cdot 6\cdot 4\cdot 3\cdot 2}$$

etc.

Sol. is

$$y = C\left[1 + \frac{x^2}{2} - \frac{x^4}{4\cdot 2} + \frac{7x^6}{6\cdot 5\cdot 4\cdot 2} - \frac{11\cdot 7x^8}{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 2} + - \cdots\right]$$

$$+ D\left[x - \frac{x^3}{3\cdot 2} + \frac{x^5}{4\cdot 3\cdot 2} - \frac{9x^7}{7\cdot 6\cdot 4\cdot 3\cdot 2} + \frac{13\cdot 9x^9}{9\cdot 8\cdot 7\cdot 6\cdot 4\cdot 3\cdot 2} - + \cdots\right].$$

e) $(1+x^2)y'' + xy' + y = 0$

$$y'' + \frac{x}{1+x^2}y' + \frac{1}{1+x^2}y = 0$$

$p(x) = \frac{x}{1+x^2}$ is analytic at 0

$q(x) = \frac{1}{1+x^2}$ is analytic at 0

So 0 is an ordinary point.

Guess $y = \sum_{j=0}^{\infty} a_j x^j$

$$y' = \sum_{j=1}^{\infty} j a_j x^{j-1}$$

$$y'' = \sum_{j=2}^{\infty} j(j-1) a_j x^{j-2}$$

$$(1+x^2)\sum_{j=2}^{\infty} j(j-1)a_j x^{j-2} + x\sum_{j=1}^{\infty} j a_j x^{j-1} + \sum_{j=0}^{\infty} a_j x^j = 0$$

$$\sum_{j=2}^{\infty} j(j-1)a_j x^{j-2} + \sum_{j=2}^{\infty} j(j-1)a_j x^j + \sum_{j=1}^{\infty} j a_j x^j + \sum_{j=0}^{\infty} a_j x^j = 0$$

$$\sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2} x^j + \sum_{j=2}^{\infty} j(j-1)a_j x^j + \sum_{j=1}^{\infty} j a_j x^j + \sum_{j=0}^{\infty} a_j x^j = 0$$

$$2\cdot 1\cdot a_2 x^0 + 3\cdot 2\cdot a_3 x + 1\cdot a_1 x + a_0 x^0 + a_1 x$$

$$+ \sum_{j=2}^{\infty} \left[(j+2)(j+1) a_{j+2} + \left(j(j-1) a_j + j a_j + a_j \right) \right] x^j = 0$$

$$\therefore \quad 2a_2 + a_0 = 0$$

$$6a_3 + a_1 + a_1 = 0$$

$$(j+2)(j+1) a_{j+2} + (j^2+1) a_j = 0, \quad j \geq 2$$

$$(a_{j+2}) = -\frac{(j^2+1) a_j}{(j+2)(j+1)}, \quad j \geq 2.$$

$$a_0 = C$$

$$a_1 = D$$

$$a_2 = -\frac{a_0}{2} = -\frac{C}{2}$$

$$a_3 = -\frac{a_1}{3} = -\frac{D}{3}$$

$$a_4 = -\frac{5a_2}{4\cdot 3} = \frac{5C}{4\cdot 3\cdot 2}$$

$$a_5 = \frac{-10a_3}{5\cdot 4} = \frac{10D}{5\cdot 4\cdot 3}$$

$$a_6 = -\frac{17a_4}{6\cdot 5} = -\frac{17\cdot 5C}{6\cdot 5\cdot 4\cdot 3\cdot 2}$$

$$a_7 = -\frac{26a_5}{7\cdot 6} = \frac{-26\cdot 10D}{7\cdot 6\cdot 5\cdot 4\cdot 3}$$

$$a_8 = \frac{-37a_6}{8\cdot 7} = \frac{37\cdot 17\cdot 5C}{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}.$$

So solution is

$$y = C\left[1 - \frac{x^2}{2} + \frac{5x^4}{4\cdot3\cdot2} - \frac{17\cdot5\,x^6}{6\cdot5\cdot4\cdot3\cdot2} + \frac{37\cdot17\cdot5\,x^8}{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2} - + \cdots\right]$$

$$+ D\left[x - \frac{x^3}{3} + \frac{10x^5}{5\cdot4\cdot3} - \frac{26\cdot10\,x^7}{7\cdot6\cdot5\cdot4\cdot3} + - \cdots\right].$$

3a) $y'' + xy' + y = 0$

$$y = \sum_{j=0}^{\infty} a_j x^j$$

$$y' = \sum_{j=1}^{\infty} j a_j x^{j-1}$$

$$y'' = \sum_{j=2}^{\infty} j(j-1) a_j x^{j-2}$$

$$\sum_{j=2}^{\infty} j(j-1) a_j x^{j-2} + x\sum_{j=1}^{\infty} j a_j x^{j-1} + \sum_{j=0}^{\infty} a_j x^j = 0$$

$$\sum_{j=2}^{\infty} j(j-1) a_j x^{j-2} + \sum_{j=1}^{\infty} j a_j x^j + \sum_{j=0}^{\infty} a_j x^j = 0$$

$$\sum_{j=0}^{\infty} (j+2)(j+1) a_{j+2} x^j + \sum_{j=1}^{\infty} j a_j x^j + \sum_{j=0}^{\infty} a_j x^j = 0$$

$$2\cdot1\cdot a_2 x^0 + a_0 x^0 + \sum_{j=1}^{\infty}\left[(j+2)(j+1) a_{j+2} + (j+1) a_j\right] x^j = 0$$

$$\therefore \quad 2a_2 + a_0 = 0$$

$$a_{j+2} = -\frac{a_j}{j+2}, \quad j \geq 1$$

$$a_0 = C$$

$$a_1 = D$$

$$a_2 = -\frac{a_0}{2} = -\frac{C}{2}$$

$$a_3 = -\frac{a_1}{3} = -\frac{D}{3}$$

$$a_4 = -\frac{a_2}{4} = \frac{+C}{4\cdot 2}$$

$$a_5 = -\frac{a_3}{5} = \frac{D}{5\cdot 3}$$

$$a_6 = -\frac{a_4}{6} = -\frac{C}{6\cdot 4\cdot 2}$$

$$a_7 = -\frac{a_5}{7} = -\frac{D}{7\cdot 5\cdot 3}$$

$$a_8 = -\frac{a_6}{8} = \frac{C}{8\cdot 6\cdot 4\cdot 2}$$

etc.

Solution is

$$y = C\left[1 - \frac{x^2}{2} + \frac{x^4}{4\cdot 2} - \frac{x^6}{6\cdot 4\cdot 2} + \frac{x^8}{8\cdot 6\cdot 4\cdot 2} - + \cdots\right]$$

$$+ D\left[x - \frac{x^3}{3} + \frac{x^5}{5\cdot 3} - \frac{x^7}{7\cdot 5\cdot 3} + \cdots\right]$$

b) Apply ratio test to first series:

$$\lim_{j\to\infty}\left|\frac{x^{2j+2}/((2j+2)(2j)(2j-2)\cdots 2)}{x^{2j}/2j(2j-2)\cdots}\right|$$

$$= \lim_{j\to\infty}\left|\frac{x^2}{2j+2}\right| = 0 < 1 \quad \forall x.$$ So series converges for all x.

Apply ratio test to second series:

$$\lim_{j\to\infty}\left|\frac{x^{2j+1}/(2j+1)(2j-1)\cdots 3}{x^{2j-1}/(2j-1)(2j-3)\cdots 3}\right| = \lim_{j\to\infty}\left|\frac{x^2}{(2j+1)}\right| = 0 < 1$$

so series converges for all x

c) $$e^{-x^2/2} = \sum_{j=0}^{\infty}\frac{(-x^2/2)^j}{j!} = \sum_{j=0}^{\infty}\frac{(-1)^j x^{2j}}{2^j\cdot j!}$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{4\cdot 2} - \frac{x^6}{6\cdot 4\cdot 2} + \cdots.$$

This is the first series.

$$v = \int \frac{1}{y_1^2} e^{-\int p(x)dx}\,dx$$

$$= \int \frac{1}{(e^{-x^2/2})^2}\cdot e^{-\int x\,dx}\,dx$$

$$= \int \frac{1}{e^{-x^2}}\cdot e^{-\frac{x^2}{2}}\,dx$$

$= \int e^{x^2/2}\,dx$. Cannot evaluate this integral

Anyway, second solution is

$y_2 = v\cdot y_1$.

(A)

Section 6.2

2. $f(x) = \begin{cases} \pi & \text{if } -\pi \le x \le \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} < x \le \pi \end{cases}$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi/2} \pi\, dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} 0\, dx$$

$$= \frac{1}{\pi} \cdot \frac{3\pi}{2} \cdot \pi = \frac{3\pi}{2}.$$

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi/2} \pi \cos jx\, dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} 0 \cdot \cos jx\, dx$$

$$= \int_{-\pi}^{\pi/2} \cos jx\, dx = \frac{\sin jx}{j}\Big]_{-\pi}^{\pi/2} = \frac{\sin j\frac{\pi}{2}}{j}$$

$$a_{2l} = \frac{\sin l\pi}{2l} = 0 \quad \text{for } l = 1, 2, \ldots$$

$$a_{2l+1} = \frac{(-1)^l}{2l+1} \quad \text{for } l = 0, 1, 2, \ldots$$

$$b_j = \frac{1}{\pi}\int_{-\pi}^{\pi/2} \pi \sin jx\, dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} 0 \cdot \sin jx\, dx$$

$$= \int_{-\pi}^{\pi/2} \sin jx\, dx = -\frac{\cos jx}{j}\Big]_{-\pi}^{\pi/2} = \frac{-\cos\frac{j\pi}{2}}{j} + \frac{\cos j\pi}{j}$$

$$b_{2l} = \frac{-\cos l\pi}{2l} + \frac{\cos 2l\pi}{2l} = \frac{(-1)^{l+1}}{2l} + \frac{1}{2l} = \frac{1+(-1)^{l+1}}{2l}, \; l \ge 1$$

$$b_{2l+1} = \frac{-\cos\frac{(2l+1)\pi}{2}}{2l+1} + \frac{\cos(2l+1)\pi}{2l+1} = \frac{(-1)^{2l+1}}{2l+1} = \frac{-1}{2l+1}, \; l \ge 0$$

So Fourier series is

$$\frac{1}{2}\cdot\frac{3\pi}{2} + \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1}\cos(2l+1)x + \sum_{l=1}^{\infty} \frac{1+(-1)^{l+1}}{2l}\sin 2l$$

$$+ \sum_{l=0}^{\infty} \frac{-1}{2l+1}\sin(2l+1)x$$

2. $$f(x) = \begin{cases} 0 & \text{if } -\pi \le x < 0 \\ 1 & \text{if } 0 \le x \le \pi/2 \\ 0 & \text{if } \frac{\pi}{2} < x \le \pi. \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi/2} 1\,dx = \frac{1}{2}$$

$$a_j = \frac{1}{\pi}\int_0^{\pi/2} 1\cdot\cos jx\,dx = \frac{1}{\pi}\cdot\frac{+\sin jx}{j}\Big]_0^{\pi/2}$$

$$= \frac{1}{\pi}\,\frac{\sin\frac{j\pi}{2}}{j}$$

$$a_{2l} = \frac{\frac{1}{\pi}\sin l\pi}{2l} = 0, \quad l = 1, 2, \ldots$$

$$a_{2l+1} = \frac{\frac{1}{\pi}\sin\frac{(2l+1)}{2}\pi}{2l+1} = \frac{1}{\pi}\,\frac{(-1)^l}{2l+1}, \quad l \ge 0.$$

$$b_j = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} 1\cdot\sin jx\,dx = \frac{1}{\pi}\cdot\frac{-\cos jx}{j}\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\pi}\left(\frac{-\cos\frac{j\pi}{2}}{j} + \frac{1}{j}\right)$$

$$b_{2l} = \frac{1}{\pi}\left(\frac{-\cos l\pi}{2l} + \frac{1}{2l}\right)$$

$$= \frac{1}{\pi}\left(\frac{(-1)^{l+1}+1}{2l}\right), \quad l \geq 1$$

$$b_{2l+1} = \frac{1}{\pi}\left(\frac{-\cos\frac{2l+1}{2}\pi}{2l+1} + \frac{1}{2l+1}\right) = \frac{1}{(2l+1)\pi}, \quad l \geq 0.$$

So Fourier series is

$$\frac{1}{2}, \frac{1}{2} + \sum_{l=0}^{\infty} \frac{1}{\pi}\frac{(-1)^l}{2l+1}\cos(2l+1)x$$

$$+ \sum_{l=1}^{\infty} \frac{1}{\pi}\left(\frac{(-1)^{l+1}+1}{2l}\right)\sin 2lx$$

$$+ \sum_{l=0}^{\infty} \frac{1}{(2l+1)\pi}\sin(2l+1)x.$$

3. $$f(x) = \begin{cases} 0, & -\pi \leq x < 0 \\ \sin x, & 0 \leq x \leq \pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = \frac{1}{\pi}\left(-\cos x\right)_0^{\pi} = \frac{1}{\pi}(1+1) = \frac{2}{\pi}.$$

$$a_j = \frac{1}{\pi}\int_0^{\pi} \sin x \cos jx\,dx = \frac{1}{\pi}\int_0^{\pi} \frac{\sin(j+1)x - \sin(j-1)x}{2}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{-\cos(j+1)x}{j+1} + \frac{\cos(j-1)x}{j-1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left(\frac{-\cos(j+1)\pi}{j+1} + \frac{\cos(j-1)\pi}{j-1}\right) + \frac{1}{2\pi}\left(+\frac{1}{j+1} - \frac{1}{j-1}\right)$$

$$= \frac{1}{2\pi}\left(\frac{(-1)^j}{j+1} - \frac{(-1)^j}{j-1}\right) + \frac{1}{2\pi}\left(\frac{1}{j+1} - \frac{1}{j-1}\right)$$

$$= \frac{1}{2\pi} \cdot \frac{1}{j^2-1}(-2)\left((-1)^j+1\right), \quad j \geq 2$$

For $j = 1$,

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \cos x \, dx = \frac{1}{2\pi}\int_0^{\pi} \sin 2x \, dx$$

$$= \frac{1}{2\pi}\left(\frac{-\cos 2x}{2}\right)_0^{\pi}$$

$$= \frac{1}{2\pi}\left(-\frac{1}{2}+\frac{1}{2}\right) = 0.$$

$$b_j = \frac{1}{\pi}\int_0^{\pi} \sin x \sin jx \, dx = \frac{1}{\pi}\int_0^{\pi} \frac{\cos(1-j)x - \cos(1+j)x}{2} dx$$

$$= \frac{1}{2\pi}\left(\frac{\sin(1-j)x}{1-j} - \frac{\sin(1+j)x}{1+j}\right)_0^{\pi}$$

$$= 0, \quad j \geq 2$$

For $j = 1$,

$$\frac{1}{\pi}\int_0^{\pi} \sin x \sin x \, dx = \frac{1}{\pi}\int_0^{\pi} \sin^2 x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{1-\cos 2x}{2} dx = \frac{1}{\pi}\left(\frac{x}{2} - \frac{\sin 2x}{4}\right)_0^{\pi}$$

$$= \frac{1}{\pi} \cdot \frac{\pi}{2} = \frac{1}{2}$$

So Fourier series is

$$\frac{1}{2} \cdot \frac{2}{\pi} + 0 \cdot \cos 1x + \sum_{j=2}^{\infty} \frac{1}{2\pi} \cdot \frac{-2}{j^2-1}\left((-1)^j+1\right)\cos jx$$

$$+ \frac{1}{2}\sin x.$$

E

5 a) $f(x) = \pi$

$a_0 = 2\pi$

all other a_j, b_j are 0.

F.S. is π

c) $f(x) = \cos x$

$a_1 = 1$

all other a_j, b_j are 0

F.S. is $\cos x$

6 a) $f(x) = \begin{cases} -a, & -\pi \le x < 0 \\ a, & 0 \le x \le \pi \end{cases}$

All a_j are 0.

$$b_j = \frac{1}{\pi}\int_{-\pi}^{0} -a \sin jx \, dx + \frac{1}{\pi}\int_{0}^{\pi} a \sin jx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} a \sin jx \, dx = \frac{2a}{\pi} \cdot \left. \frac{-\cos jx}{j} \right]_0^{\pi}$$

$$= \frac{2a}{\pi}\left[\frac{-\cos j\pi + \cos 0}{j}\right]$$

$$= \frac{2a}{\pi}\left[\frac{(-1)^{j+1} + 1}{j}\right].$$

F.S. is

$$\sum_{j=1}^{\infty} \frac{2a}{\pi}\left(\frac{(-1)^{j+1}+1}{j}\right) \sin jx.$$

F

c) $f(x) = \begin{cases} -\pi/4, & -\pi \le x < 0 \\ \pi/4, & 0 \le x \le \pi \end{cases}$

This is the special case of (d) with $a = \pi/4$.

So

F.S. is

$$\sum_{j=1}^{\infty} \frac{1}{2}\left(\frac{(-1)^{j+1}+1}{j}\right)\sin jx.$$

Section 6.2

2. $f(x) = \begin{cases} -\pi & \text{if } -\pi \le x < 0 \\ x & \text{if } 0 \le x \le \pi \end{cases}$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} -\pi\, dx + \frac{1}{\pi}\int_{0}^{\pi} x\, dx$$

$$= \frac{1}{\pi}(-\pi)\pi + \frac{1}{\pi}\cdot\frac{\pi^2}{2} = -\pi + \frac{\pi}{2} = -\frac{\pi}{2}.$$

$$a_j = \frac{1}{\pi}\int_{-\pi}^{0} -\pi \cos jx\, dx$$

$$+ \frac{1}{\pi}\int_{0}^{\pi} x\cos jx\, dx$$

$$= -\frac{\sin jx}{j}\Big]_{-\pi}^{0} + \frac{1}{\pi}\left(\frac{x\sin jx}{j} + \frac{\cos jx}{j^2}\right)\Big]_{0}^{\pi}$$

$$= (-0+0) + \frac{1}{\pi}\left[\left(\frac{\pi\sin j\pi}{j} + \frac{\cos j\pi}{j^2}\right) - \left(\frac{0\sin 0}{j} + \frac{\cos 0}{j^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^j}{j^2} - \frac{1}{j^2}\right] = \frac{1}{\pi}\cdot\frac{(-1)^j - 1}{j^2}.$$

$$b_j = \frac{1}{\pi}\int_{-\pi}^{0} -\pi \sin jx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin jx\,dx$$

$$= \left.\frac{\cos jx}{j}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{-x\cos jx}{j} + \frac{\sin jx}{j^2}\right]_{0}^{\pi}$$

$$= \frac{\cos 0}{j} - \frac{\cos j\pi}{j} + \frac{1}{\pi}\left[\frac{-\pi\cos j\pi}{j} + \frac{0}{j^2} + \frac{0}{j} - \frac{0}{j^2}\right]$$

$$= \frac{1-(-1)^j}{j} - \frac{(-1)^j}{j} = \frac{1+2(-1)^{j+1}}{j}$$

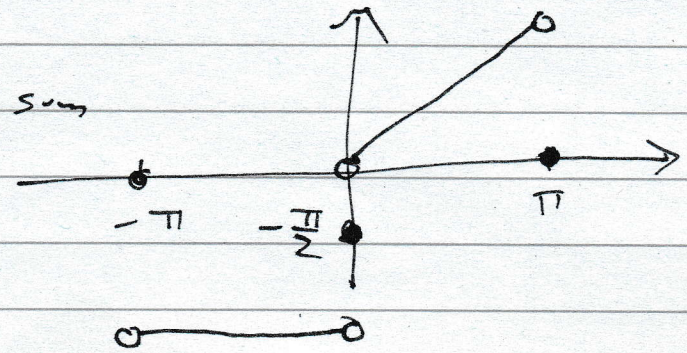
F.S. is

$$\frac{1}{2}\left(-\frac{\pi}{2}\right) + \sum_{j=1}^{\infty}\frac{1}{\pi}\frac{(-1)^j-1}{j^2}\cos jx + \sum_{j=1}^{\infty}\frac{1+2(-1)^{j+1}}{j}\sin jx$$

The left limit of f at 0 is $-\pi$.
The right limit of f at 0 is 0.
The average of these is $-\frac{\pi}{2}$. The F.S. will, by Dirichlet's theorem, converge to $-\frac{\pi}{2}$ at 0.

The left limit at $\pi \leftrightarrow -\pi$ is π. The right limit is $-\pi$, the average is 0. The F.S. will converge to 0 at π & $-\pi$.

graph of sum of Fourier series



4. Dirichlet's theorem applies. The F.S. converges to the average of the left- and right-limits. This is 0.