FIRST MIDTERM EXAM

**General Instructions:** Read the statement of each problem carefully. On the multiple choice questions you need not show your work. On the two short answer questions you definitely do need to show your work if you want to receive full credit.

Be sure to ask questions if anything is unclear. This exam is worth 100 points. You will have two hours to take this exam.
(8 points) 1. The differential equation \( y' = xy^2 \) has general solution given by

(a) \( y = \frac{2}{C - x^2} \)
(b) \( y = \frac{2}{(C + x^2)^2} \)
(c) \( y = \frac{2}{(C + x^4)^2} \)
(d) \( y = \frac{4}{(C - x^2)^2} \)
(e) \( y = \frac{1}{(C - x^2)^2} \)

\[
\frac{dy}{dx} = xy^2
\]

\[
\frac{dy}{y^2} = x \, dx
\]

\[
-\frac{1}{y} = \frac{x^2}{2} + E
\]

\[
\frac{1}{y} = D - \frac{x^2}{2}
\]

\[
y = \frac{1}{D - x^2/2} = \frac{2}{C - x^2}
\]
2. Solve the first-order linear equation

\[ y' + 2x \cdot y = 3x. \]

(a) \( \frac{1}{2} + Ce^{-x^2} \)
(b) \( \frac{3}{2} + Ce^{-2x^2} \)
(c) \( \frac{3}{2} + Ce^{-x^2} \)
(d) \( -\frac{1}{2} + Ce^{-x^2} \)
(e) \( Ce^{-x^2} \)

\[
\int e^{(x^2)} dx = \int 2x \cdot dx = \frac{x^2}{2}
\]

\[
e^{x^2} \cdot y' + e^{x^2} \cdot 2x \cdot y = 3x \cdot e^{x^2}
\]

\[
(e^{x^2} \cdot y)' = 3x \cdot e^{x^2}
\]

\[
\int (e^{x^2} \cdot y)' \cdot dx = \int 3x \cdot e^{x^2} \cdot dx
\]

\[
e^{x^2} \cdot y = \frac{3}{2} \cdot x \cdot e^{x^2} + C
\]

\[
y = \frac{3}{2} + Ce^{-x^2}
\]
(10 points) 3. The solution of the initial value problem

\[ y' + (1/x)y = x^2 \]
\[ y(1) = 2 \]

is what?

(a) \( y = x^3/2 + 7/(4x) \)
(b) \( y = x^3/4 + 7/(4x) \)
(c) \( y = x^3 + 7/(4x) \)
(d) \( y = x^3/4 + 3/(4x) \)
(e) \( y = x^3/4 - 7/(4x) \)

\[
\int_{x_0}^{x} e \, dx = e^{\frac{1}{x}} \int_{x_0}^{x} e \, dx = e \ln x = 1 \cdot x
\]

For \( x > 0 \),

\[
x y' + y = x^3
\]
\[
(xy)' = x^3
\]
\[
\int (xy)' \, dx = \int x^3 \, dx
\]
\[
xy = \frac{x^4}{4} + C
\]
\[
y = \frac{x^3}{4} + \frac{C}{x}
\]
\[
2 = y(1) = \frac{1}{4} + C \Rightarrow C = \frac{7}{4}
\]

So, particular solution is \( y = \frac{x^3}{4} + \frac{7}{4x} \).
(8 points) 4. Solve the exact equation

\[(x^2 - y)dx + (y^2 - x)dy = 0.\]

(a) \(x^3 + 3xy + y^3 = C\)
(b) \(x^3 - 2xy + y^3 = C\)
(c) \(x^3 + 2xy + y^3 = C\)
(d) \(x^3 - xy + y^3 = C\)
(e) \(x^3 - 3xy + y^3 = C\)

\[M(x,y) = x^2 - y \quad N(x,y) = y^2 - x\]

\[\frac{\partial M}{\partial y} = -1 \quad \frac{\partial N}{\partial x} = -1, \text{ So equation is exact.}\]

\[\frac{\partial f}{\partial x} = x^2 - y \implies f(x,y) = \frac{x^3}{3} - xy + \varphi(y)\]

\[y^2 - x = N = \frac{\partial f}{\partial y} = -x + \varphi'(y)\]

\[\varphi'(y) = y^2 \implies \varphi(y) = \frac{y^3}{3} + D\]

\[f(x,y) = \frac{x^3}{3} - xy + \frac{y^3}{3} + D\]

Solution to ODE is \[\frac{x^3}{3} - xy + \frac{y^3}{3} = E\]

or \[x^3 - 3xy + y^3 = C.\]
(8 points) 5. Find the family of orthogonal trajectories to the curves \( y = cx^2 \).

(a) \( y = \sqrt{C + x^2} \)
(b) \( y = \sqrt{C}x^2 \)
(c) \( y = \sqrt{C - 2x^2} \)
(d) \( y = \sqrt{C - x^2/2} \)
(e) \( y = \sqrt{C - x^3/2} \)

\[
\frac{dy}{dx} = c \\
\frac{y}{x^2} = C \\
x^2y' - y \cdot 2x = 0 \\
\frac{x^2y' - y \cdot 2x}{x^4} = 0 \\
x^2y' = 2y \\
y' = \frac{2y}{x} \\
\text{Or, orthogonal trajectory satisfies} \\
y' = -\frac{x}{2y} \\
2y dy = -xdx \\
y^2 = -\frac{x^2}{2} + C \\
y = \sqrt{C - \frac{x^2}{2}}
\]
(8 points) 6. Use the method of homogeneous equations to solve

\[(x - y)dx + (x + y)dy = 0.\]

(a) \(\frac{1}{3}\ln(1 + y^2/x^2) + \arctan(y/x) + \ln x = C\)

(b) \(\frac{1}{2}\ln(1 + y^2/x^2) + \arctan(y/x) - \ln x = C\)

(c) \(\frac{1}{2}\ln(1 + y^2/x^2) + \arctan(y/x) + \ln x = C\)

(d) \(\frac{1}{2}\ln(1 + y^2/x^2) + \arctan(y/x) + 2\ln x = C\)

(e) \(-\frac{1}{2}\ln(1 + y^2/x^2) + \arctan(y/x) + \ln x = C\)

\[
(x-y)\,dx + (x+y)\,dy = 0
\]

\[
\frac{dy}{dx} = \frac{y-x}{y+x}
\]

\[
\frac{dy}{dx} = \frac{y+1}{x+1}
\]

Set \(z = \frac{y}{x}\) \(\Rightarrow \frac{dy}{dx} = z + x \cdot \frac{dz}{dx}\)

\[
z + x \frac{dz}{dx} = \frac{z-1}{z+1}
\]

\[
x \frac{dz}{dx} = \frac{z-1}{z+1} - z = -\frac{1-z^2}{z+1}
\]

\[
\frac{z+1}{z+z^2} \,dz = -\frac{dx}{x}
\]

\[
\int \frac{z}{z^2+1} \,dz + \int \frac{1}{1+z^2} \,dz = -\int \frac{dx}{x}
\]

\[
\frac{1}{2} \log (1+z^2) + \arctan (y/x) + \ln x = C, \quad x > 0.
\]
(8 points) 7. The equation

$$(xy - 3x^3)\,dx + x^2\,dy = 0$$

is not exact. Find an integrating factor for this equation.

(a) $1/x$
(b) $1/x^2$
(c) $x$
(d) $x^2$
(e) $x^3$

$$g(x) = \frac{\partial M/\partial y - \partial N/\partial x}{N} = \frac{x - 2x}{x^2} = \frac{-x}{x^2} = -\frac{1}{x}$$

$$\mu(x) = e^{\int g(x)\,dx} = e^{-\frac{1}{x}} \cdot x = \frac{1}{x}$$

For $x > 0$, we multiply the ODE by $\frac{1}{x}$.

$$(y - 3x^2)\,dx + x\,dy = 0$$

Now $\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = 0$, so this is exact.
(8 points) 8. Use the method of reduction of order to solve the differential equation

\[ xy'' + y' = 4x. \]

(a) \( y = x^3 + C \ln x + D \)
(b) \( y = x + C \ln x + D \)
(c) \( y = x^4 + C \ln x + D \)
(d) \( y = x + C \ln x \)
(e) \( y = x^2 + C \ln x + D \)

\[
\begin{align*}
y' &= \rho \\
y'' &= \rho'
\end{align*}
\]

\[ x\rho' + \rho = 4x \]

\[ \rho' + \frac{1}{x} \rho = 4 \]

\[ e^{\int \frac{1}{x} \, dx} = e^{\ln|x|} = 1 \cdot 1. \]

For \( x > 0 \), get

\[ x\rho' + \rho = 4x \]

\[ (x\rho)' = 4x \]

\[ \int (x\rho)' \, dx = \int 4x \, dx \quad \Rightarrow \quad x\rho = 2x^2 + C \]

\[ \rho = 2x + \frac{C}{x}. \]

\[ \frac{dy}{dx} = 2x + \frac{C}{x} \]

\[ y = x^2 + C \ln x + D \]

For \( x > 0 \), this is \( y = x^2 + C \ln x + D \)
(10 points) 9. Find the solution of the exact equation

\[ 2xy dx + x^2 dy = 0 \]

that satisfies \( y = 1 \) when \( x = 2 \).

(a) \( y = \frac{8}{x^3} \)
(b) \( y = \frac{4}{x^2} \)
(c) \( y = \frac{2}{x^2} \)
(d) \( y = \frac{1}{x^2} \)
(e) \( y = \frac{16}{x^4} \)

\[
M = 2xy, \quad N = x^2
\]

\[
\frac{2M}{\partial y} = 2x, \quad \frac{2N}{\partial x} = 2x \quad \text{so equation is exact.}
\]

\[
\frac{\partial F}{\partial x} = 2xy \quad \Rightarrow f(x,y) = x^2y + g(y)
\]

\[
x^2 = N = \frac{\partial F}{\partial y} = x^2 + \varphi'(y) \quad \Rightarrow \varphi'(y) = 0 \quad \Rightarrow \varphi(y) = C.
\]

So, \( f(x,y) = x^2y + C \).

Solution to \( \partial x \in I \)

\[
x^2y = D
\]

\[
y = \frac{D}{x^2}
\]

\[
z = y(2) = \frac{D}{2^2} \quad \Rightarrow \quad D = 4.
\]

Particular solution is \( y = \frac{4}{x^2} \).
(8 points) 10. Use reduction of order to solve the differential equation

\[ y'' = 2y' \]

with initial conditions \( y(0) = 0 \) and \( y'(0) = 1 \).

(a) \( y = (1/3)(-1 + e^{2x}) \)
(b) \( y = (1/2)(-2 + e^{2x}) \)
(c) \( y = (1/2)(-1 + 2e^{2x}) \)
(d) \( y = (1/4)(-1 + e^{2x}) \)
(e) \( y = (1/2)(-1 + e^{2x}) \)

\[ y' = \rho, \quad y'' = \frac{d\rho}{dy} \cdot \rho \]

\[ \frac{d\rho}{dy} \cdot \rho = 2 \rho \]

\[ \frac{d\rho}{dy} = 2 \]

\[ \rho = 2y + C \]

\[ y' = 2y + C \]

\[ y' - 2y = C \]

\[ e^{\int (1) \, dx} = e^{\int 2 \, dx} = e^{2x} \cdot S_0 \]

\[ e^{-2x} y' - 2e^{-2x} y = Ce^{-2x} \]

\[ (e^{-2x} y)' = Ce^{-2x} \]

\[ e^{-2x} y = -\frac{e}{2} e^{-2x} + D \]

\[ y = -\frac{e}{2} + De^{-2x} \]

\[ y' = 2De^{-2x} \]

\[ 0 = y(0) = -\frac{e}{2} + D \]

\[ 1 = y'(0) = 2D \]

\[ \Rightarrow C = 1, \quad D = \frac{1}{2} \]

\[ y = -\frac{1}{2} + \frac{1}{2} e^{2x} \]
11. Solve the initial value problem

\[ y' + xy = x, \quad y(0) = 1. \]

\[
\begin{align*}
\int 2x \, dx &= \int x \, dx = e^{\frac{x^2}{2}} \\
\frac{e^{\frac{x^2}{2}}}{y} &= xe^{\frac{x^2}{2}} \\
(e^{\frac{x^2}{2}}) &= xe^{\frac{x^2}{2}} + C \\
e^{\frac{x^2}{2}}y &= e^{\frac{x^2}{2}} + C \\
y &= 1 + Ce^{-\frac{x^2}{2}} \\
1 &= y(0) = 1 + C \quad \Rightarrow \quad C = 0.
\end{align*}
\]

So the solution of the IVP is \( y = 1 \).
(8 points) 12. Solve the differential equation

\[ y' = xy^3. \]

\[
\frac{dy}{dx} = xy^3
\]

\[
\frac{dy}{y^3} = x \, dx
\]

\[-\frac{1}{2} y^{-2} = \frac{x^2}{2} + C
\]

\[ y^{-2} = -x^2 + D
\]

\[ y^2 = \frac{1}{D - x^2}
\]

\[ y = \sqrt{\frac{1}{D - x^2}}. \]